GKZ HYPERGEOMETRIC SYSTEMS AND TORIC MIRROR SYMMETRY

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ABSTRACT OF THE DISSERTATION

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In this dissertation, we study questions related to the Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric systems and toric mirror symmetry. Such systems are conjecturally the de-categorification of an isotrivial family of triangulated categories over the stringy Kähler moduli space associated to a toric Gorenstein singularity, which underlies the derived equivalences between different crepant resolutions of this singularity.

The two main results of this dissertation are the following:

(1) We study the relationship between solutions to such systems near different large radius limit points, and their geometric counterparts given by the K-groups of the corresponding crepant resolutions. We prove that analytic continuation transformations of solutions are realized by the natural Fourier-Mukai transforms associated to the toric wall-crossings. This settles a conjecture of Borisov and Horja [8].

(2) We apply such systems to study toric Calabi-Yau Deligne-Mumford stacks and their Hori-Vafa mirrors. We verify that A- and B-model integral structures coincide by establishing the equality between A-brane and B-brane central charges, in terms of period integrals and Gamma series respectively. This settles a (variation of a) conjecture of Hosono [19].

This dissertation builds upon the papers [17] and [16] of the author.

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Dedicated to Ludwig van Beethoven,
whose immortal music and transcendent spirit
have always been –
and will always be –
a profound source of inspiration.

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Chapter 1

Introduction

Mirror symmetry is a phenomenon originally observed by physicists [12] in the early 1990s, which, roughly speaking, describes a relationship between the algebraic geometry of a space and the symplectic geometry of its mirror space. The first example of mirror symmetry is the observation that one can count rational curves on a smooth quintic threefold in $\mathbb{CP}^4$ by looking at an ODE (the Picard-Fuchs equation) coming from a mirror of the quintic threefold.

Since then, there have been various mathematical formulations of mirror symmetry. One of the most remarkable formulations is the homological mirror symmetry proposed by Kontsevich in 1994. The statement, roughly speaking, says that the derived category of coherent sheaves on a Calabi-Yau manifold should be equivalent to the derived Fukaya category of the mirror manifold. Homological mirror symmetry has been the most active area in the study of mirror symmetry.

Apart from attempts to formulate the phenomenon of mirror symmetry in a mathematically rigorous way, another direction in the study of mirror symmetry is to construct explicit examples of mirror pairs. In this direction, the most prolific construction is the Batyrev-Borisov mirror construction, which allows one to start with certain combinatorial datum and construct mirror pairs as Calabi-Yau complete intersections in toric varieties. The area of the study of such mirror pairs is called toric mirror symmetry.

Now we explain the motivation behind this project. We consider a Gorenstein toric variety $X = \operatorname{Spec} \mathbb{C}[C^\vee \cap N]$ associated to a rational polyhedral cone C in a lattice N. If we take a simplicial subdivision Σ of C whose ray generators lie on a hyperplane, then the corresponding toric stack $\mathbb{P}_{\boldsymbol{\Sigma}}$ is a crepant resolution of X. According to a result of Kawamata [25], all the $\mathbb{P}_{\boldsymbol{\Sigma}}$'s are derived-equivalent.

However, there is no canonical equivalence between derived categories of different crepant resolutions. This fact suggests that instead of a discrete set of equivalences between derived categories, there should exist a continuous family of triangulated categories over a certain moduli space, the *stringy Kähler moduli space*. Roughly speaking, the stringy Kähler moduli space is the space of Kähler structures that comes from the symplectic geometry of the Calabi-Yau manifold. In general there is no global definition, however in the toric case there is an explicit construction as the complement of the zero locus of a certain Laurent polynomial (the *GKZ discriminant*) associated to the combinatorial datum. Inside the stringy Kähler moduli space there are large radius limits corresponding to crepant resolutions $\mathbb{P}_{\boldsymbol{\Sigma}}$. The fibers near the large radius limit point corresponding to $\mathbb{P}_{\boldsymbol{\Sigma}}$ are given by $D^b(\mathbb{P}_{\boldsymbol{\Sigma}})$, and the derived equivalences between two crepant resolutions are realized by paths in the stringy Kähler moduli space.

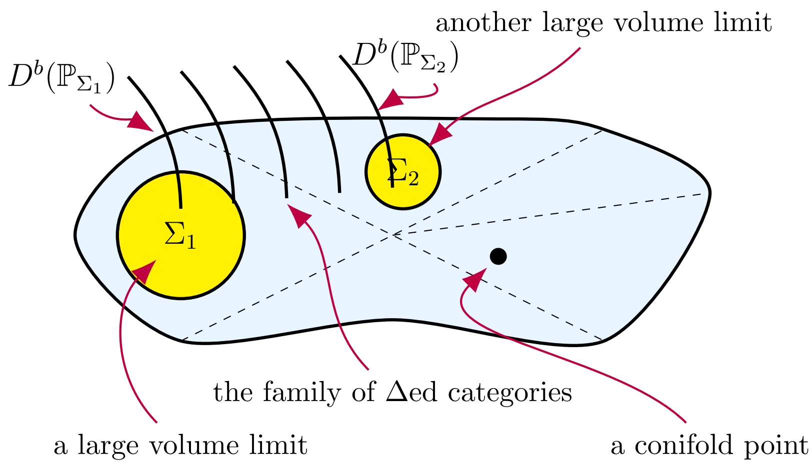

Figure 1.1: Stringy Kähler moduli space

The construction of such an isotrivial family of triangulated categories is a long-standing open problem. However, its decategorification is a local system over the moduli space that can be described as a system of linear PDEs, known as the *GKZ hypergeometric system*.

We start with our combinatorial setting. Let C be a finite rational polyhedral cone in a lattice $N = \mathbb{Z}^{\mathrm{rk}N}$. We assume that all ray generators of C lie on a primitive hyperplane $\deg(\cdot) = 1$ where $\deg : N \to \mathbb{Z}$ is a linear function (or equivalently, C is the cone over a $(\mathrm{rk}N - 1)$-dimensional polytope Δ of height 1). This data encodes an affine toric variety $X = \operatorname{Spec} \mathbb{C}[N^\vee \cap C^\vee]$, with the hyperplane condition equivalent to X being Gorenstein, i.e. having trivial dualizing sheaf.

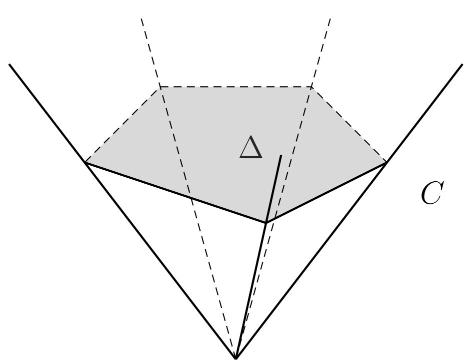

Figure 1.2: A 3-dimensional example

Let $\{v_i\}_{i=1}^n$ be a set of n lattice points in C which includes all of its ray generators, with $\deg(v_i) = 1$ for all i. One can construct (stacky) crepant resolutions $\mathbb{P}_{\mathbf{\Sigma}} \to X$, where the stacky fan $\mathbf{\Sigma}$ is obtained by subdivisions Σ of C based on triangulations that involve some of the points v_i. Note that the additional data $\{v_i\}$ in the definition of $\mathbf{\Sigma}$ is chosen to be these deg 1 points.

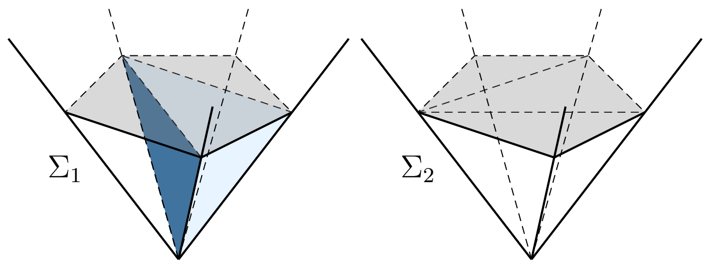

Figure 1.3: Two different triangulations

Definition 1 (better-behaved GKZ systems, [7] and [22])**.** To each lattice point c in the cone C we attach a holomorphic function $\Phi_c(x_1, \ldots, x_n)$ defined on the stringy Kähler moduli space, and consider a linear system of PDEs:

$$\mathrm{bbGKZ}(C) : \begin{cases} \partial_i \Phi_c = \Phi_{c+v_i}, \quad \forall c \in C, i = 1, \cdots, n \\ \sum_{i=1}^n \langle \mu, v_i \rangle x_i \partial_i \Phi_c + \langle \mu, c \rangle \Phi_c = 0, \quad \forall c \in C, \mu \in N^\vee \end{cases}$$

A compactly-supported version bbGKZ(C°) could be defined similarly by considering lattice points in the interior C° only.

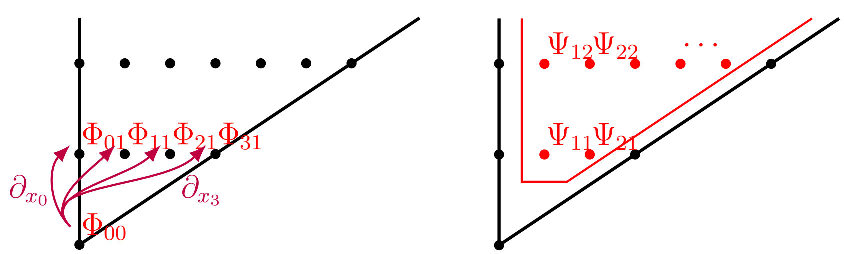

Figure 1.4: Example of $[\mathbb{C}^2/\mathbb{Z}_3]$

The key property of these systems is that their solution spaces are finite-dimensional and can be canonically identified, via certain Gamma series, with the Grothendieck groups of the derived categories $D^b(\mathbb{P}_{\mathbf{\Sigma}})$ and $D^b_c(\mathbb{P}_{\mathbf{\Sigma}})$ in a neighborhood of the large radius limit point corresponding to Σ.

In [8], Borisov and Horja proposed two conjectures on bbGKZ systems: the duality conjecture and the analytic continuation conjecture. Roughly speaking, they are concerned with how to express natural structures and operations on derived categories in terms of bbGKZ systems: the Euler pairing and Fourier-Mukai transforms. These two conjectures are settled in full generality in [5] and [17].

More precisely, the main result of [5] is an explicit construction of the pairing of GKZ systems that recovers the Euler pairing near any large radius limit points. This result can be viewed as the B-model interpretation of Iritani's Gamma integral structure [21], within the context of local mirror symmmetry.

Theorem 2 ([5]). *Let Φ and Ψ be solutions to* $\mathrm{bbGKZ}(C,0)$ *and* $\mathrm{bbGKZ}(C^\circ,0)$ *respectively. We define the GKZ pairing by the formula*

$$\langle \Phi, \Psi\rangle_{\mathrm{GKZ}} = \sum_{\substack{c\in C, d\in C^\circ \\ I\subseteq\{1,\cdots,n\}, |I|=\mathrm{rk}N}} \xi_{c,d,I}\,\mathrm{Vol}_I\left(\prod_{i\in I} x_i\right)\Phi_c\Psi_d$$

where the coefficients $\xi_{c,d,I}$ and Vol_I only depends on the combinatorics of C. Then $\langle \Phi, \Psi\rangle_{\mathrm{GKZ}}$ is a constant for any solutions Φ and Ψ. Furthermore, $\langle -, -\rangle_{\mathrm{GKZ}}$ coincide with the Euler pairing in the neighborhood of any large radius limit point.

Building on this duality result, the first main result of this dissertation is the following, which could also be seen as a version of Crepant Transformation Conjecture in the context of local mirror symmetry. This result provides evidence on why one should think of the GKZ systems as the correct de-categorification that underlies the conjectural isotrivial family of triangulated categories.

Theorem 3 (= Theorem 54, 55)**.** *The following diagrams commute:*

$$\begin{array}{ccc} K_0(\mathbb{P}_{\mathbf{\Sigma}_+})^\vee & \xrightarrow{-\circ\Gamma_+} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_+)) \\ \Big\downarrow{\scriptstyle \mathrm{FM}^\vee} & & \Big\downarrow{\scriptstyle \mathrm{MB}} \\ K_0(\mathbb{P}_{\mathbf{\Sigma}_-})^\vee & \xrightarrow{-\circ\Gamma_-} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_-)) \end{array}$$

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\mathbf{\Sigma}_+})^\vee & \xrightarrow{-\circ\Gamma_+^\circ} & Sol(\mathrm{bbGKZ}(C^\circ),U_+) \\ \Big\downarrow{\scriptstyle (\mathrm{FM}^c)^\vee} & & \Big\downarrow{\scriptstyle \mathrm{MB}^c} \\ K_0^c(\mathbb{P}_{\mathbf{\Sigma}_-})^\vee & \xrightarrow{-\circ\Gamma_-^\circ} & Sol(\mathrm{bbGKZ}(C^\circ),U_-) \end{array}$$

where the horizontal arrows are mirror symmetry maps, FM *(*FM^c*) and* MB *(*MB^c*) denote the Fourier-Mukai transforms and analytic continuation transformations of solutions respectively.*

This phenomenon was first observed in the PhD thesis of Horja [18], and was studied by Borisov and Horja in [10] later. However the authors were using the original version of the GKZ systems, and the map between the dual of the K-theory and the solution space is not necessarily an isomorphism due to the rank-jumping phenomenon at non-generic parameters (see e.g. [26]). The advantage of the bbGKZ systems is that that the mirror symmetry maps from the dual of the K-groups to the solution spaces are always isomorphisms.

The second part of this dissertation focuses on a generalization of the so-called Hori-Vafa mirror construction, based on the original definition of Hosono [19] in the 2- and 3-dimensional cases. Our B-models are toric Calabi-Yau orbifolds $\mathbb{P}_{\mathbf{\Sigma}}$, whereas we take the corresponding A-models to be Landau-Ginzburg models $((\mathbb{C}^*)^d, f)$, where $f : (\mathbb{C}^*)^d \to \mathbb{C}$ is a Laurent polynomial with fixed Newton polytope. While there are certain issues with this definition (see Remark 61), we are able to prove that the *integral structures* on the A- and B-sides coincide by showing the equality between

A- and B-branes central charges. Integral structures in mirror symmetry has been extensively studied by Iritani, especially in the context of toric mirror symmetry (Batyrev-Borisov mirrors), see e.g. [21, 22].

Theorem 4 (= Theorem 62)**.** *The A- and B-model integral structures of the Hori-Vafa mirrors, defined by* $H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$ *and* $K_0(\mathbb{P}_\Sigma, \mathbb{Z})$ *respectively, coincide.*

Remark 5. We make a final remark regarding the second main result of this dissertation. The original motivation behind this work was an (unsuccessful) attempt to construct a global integral structure for better-behaved GKZ systems. To do this, we need to identify the correct space of D-branes away from the large radius limit and construct a map from it to the solution space of bbGKZ systems. In the context of Calabi-Yau hypersurfaces (or more generally complete intersections) in toric varieties, the natural candidates are given by certain relative homology groups and oscillatory integrals (see e.g. [21] for details). In our local mirror symmetry setting, it is natural to consider the Hori-Vafa mirrors of the toric stacks $\mathbb{P}_{\mathbf{\Sigma}}$ associated to the combinatorial datum. However, technical difficulties arise when one tries to prove the convergence of the associated period integrals, which is the reason why we have to restrict ourselves to a special kind of Lagrangian submanifolds of the LG model $((\mathbb{C}^*)^d, f)$ which are defined in an *ad hoc* manner. We believe a better understanding of the homological mirror symmetry for toric Calabi-Yau orbifolds $\mathbb{P}_{\mathbf{\Sigma}}$ would help to resolve this issue. We hope to return to this problem in a future work.

This dissertation is organized as follows.

In Chapter 2 we cover the background knowledge of this dissertation. In Section 2.1 we recall the basic definition and properties of smooth toric Deligne-Mumford stacks and their twisted sectors. In Section 2.2 we define the secondary fan of toric stacks and the associated toric wall-crossings. In Section 2.3 we give combinatorial descriptions of the (usual and compactly supported) derived categories, K-theories and orbifold cohomology of toric stacks. In Section 2.4 we introduce the better-behaved GKZ systems and recall some of their important properties. In Section 2.5, we formulate the bbGKZ systems in terms of the language of D-modules, and discuss the relations between the duality result in the previous section and a recent result of Reichelt-Sevenheck-Walther.

In Chapter 3 we prove the first main result of this dissertation, namely the analytic continuation of Gamma series solutions to bbGKZ systems between adjacent large volume limits coincides with the pullback-pushforward functor associated to the toric wall-crossing of the corresponding toric stacks. In Section 3.1 we compute the analytic continuation of Gamma series solution by applying the Mellin-Barnes integral method. In Section 3.2 we give a combinatorial formula for the pullback-pushforward functor associated to the toric wall-crossing, and observe that it coincides with the computation of the previous section. In Section 3.3 we utilize the duality result on the bbGKZ systems to deduce the parallel results for the dual systems.

In Chapter 4 we prove the second main result of this dissertation, namely the coincidence between the A- and B-model integral structures, by proving the equality of the corresponding A- and B-brane central charges. In Section 4.1 we give the definitions of our (modified) version of central charges, in terms of period integrals and Gamma series respectively. In Section 4.3 we prove a technical result on the relationship between certain integral of orbifold cohomology classes on toric stacks and the volume of certain polytopes, which is essential to the computation in Section 4.2. Finally in Section 4.4 we establish the desired equality between A- and B-brane central charges.

Chapter 2

Toric Deligne-Mumford stacks and better-behaved GKZ systems

In this chapter we review the background knowledge of this dissertation. In Section 2.1 we recall the basic definition and properties of smooth toric Deligne-Mumford stacks and their twisted sectors. In Section 2.2 we define the secondary fan of toric stacks and the associated toric wall-crossings. In Section 2.3 we give combinatorial descriptions of the (usual and compactly supported) derived categories, K-theories and orbifold cohomology of toric stacks. In Section 2.4 we introduce the better-behaved GKZ systems and recall some of their important properties.

2.1 Toric Deligne-Mumford stacks

In this section, we review the construction of smooth toric Deligne-Mumford stacks from certain combinatorial data called *stacky fan*, following Borisov-Chen-Smith [9].

Definition 6. Let N be a finitely generated free[1] abelian group of rank d. Denote $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. Let Σ be a simplicial fan in the vector space $N_{\mathbb{Q}}$. We fix a choice

[1]The construction of [9] actually works for a general finitely generated abelian group, without the freeness assumption. For the sake of simplicity we will work with this additional assumption.

of $v_i \in N$ for each 1-dimensional cones (i.e., rays) ρ_i for $i = 1, \cdots, n$ of the fan Σ such that v_i generate the cone ρ_i. The set $\{b_1, \cdots, b_n\}$ defines a homomorphism $\beta : \mathbb{Z}^n \to N$ with finite cokernel. The triple $\mathbf{\Sigma} := (N, \Sigma, \beta)$ is called a *stacky fan.*

Remark 7. Sometimes we allow some v_i's to be not present in the fan Σ. Therefore strictly speaking we will be using the *extended stacky fans* introduced by Jiang [23].

Note that the element v_i needs not to be the primitive generator of the ray ρ_i. We can associate a smooth Deligne-Mumford stack to a stacky fan $\mathbf{\Sigma}$ as follows.

Theorem 8. *Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan defined as above. Consider the open subset U of $\mathbb{C}^n$ defined by*

$$U = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : \ \{i : z_i = 0\} \in \Sigma\}$$

and a subgroup G of $(\mathbb{C}^*)^n$ defined by

$$G = \left\{ (\lambda_1, \cdots, \lambda_n) : \prod_{i=1}^{n} \lambda_i^{\langle m, v_i \rangle} = 1, \forall m \in N^\vee \right\}$$

Then the stack quotient $[U/G]$ is a smooth Deligne-Mumford stack, which we denote by $\mathbb{P}_{\mathbf{\Sigma}}$. Furthermore, its coarse moduli space is the usual toric variety $\mathbb{P}_\Sigma$ associated to the non-stacky fan Σ.

Proof. See [9, Section 3]. ☐

Generalizing the usual construction of toric varieties, there exists a correspondence between certain closed substacks of $\mathbb{P}_{\mathbf{\Sigma}}$ and the cones in the stacky fan $\mathbf{\Sigma}$. More precisely, let σ be a cone in the fan Σ, and let $N(\sigma)$ be the quotient of the lattice N by the sublattice N_σ generated by the elements v_i for $\rho_i \subseteq \sigma$. This naturally induces a quotient fan Σ/σ defined as

$$\Sigma/\sigma := \{\tau + N_\sigma : \sigma \subseteq \tau \text{ and } \tau \in \Sigma\}$$

and a homomorphism $\beta(\sigma)$ defined by the image of v_i's in the quotient lattice $N(\sigma)$. These datum altogether define a quotient stacky fan $\mathbf{\Sigma}/\boldsymbol{\sigma} := (N(\sigma), \Sigma/\sigma, \beta(\sigma))$.

Proposition 9. If σ is a cone in the stacky fan $\mathbf{\Sigma}$ which satisfies certain non-degenerate condition, then the toric stack $\mathbb{P}_{\mathbf{\Sigma}/\boldsymbol{\sigma}}$ associated to the quotient stacky fan defines a closed substack of $\mathbb{P}_{\mathbf{\Sigma}}$.

Proof. See [9, Proposition 4.2]. □

The *inertia stack* $\mathcal{I}(\mathcal{X})$ of a Deligne-Mumford stack $\mathcal{X}$ is defined as the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ along the diagonal embedding $\Delta : \mathcal{X} \hookrightarrow \mathcal{X} \times \mathcal{X}$. Its connected components are called the *twisted sectors*[2] of the stack $\mathcal{X}$. Since the twisted sectors play an essential role in the definition of the orbifold Chow ring (or orbifold cohomology) of a Deligne-Mumford stack, we review the explicit combinatorial description of them in the case of smooth toric Deligne-Mumford stacks.

Definition 10. For each cone $\sigma \in \Sigma$ we define $\mathrm{Box}(\sigma)$ to be the set of lattice points γ which can be written as $\gamma = \sum_{i\in\sigma} \gamma_i v_i$ with $0 \leq \gamma_i < 1$. We denote the union of all $\mathrm{Box}(\sigma)$ by $\mathrm{Box}(\Sigma)$.

Proposition 11. The twisted sectors of $\mathbb{P}_{\mathbf{\Sigma}}$ are in 1-1 correspondence with the set $\mathrm{Box}(\Sigma)$ by $\mathbb{P}_{\mathbf{\Sigma}/\sigma(\gamma)} \leftrightarrow \gamma$, where $\sigma(\gamma)$ denotes the minimal cone in Σ that contains γ.

Proof. See [9, Proposition 4.7]. □

Definition 12. We define the *dual* of a twisted sector $\gamma = \sum \gamma_i v_i$ by

$$\gamma^\vee = \sum_{\gamma_i \neq 0} (1 - \gamma_i) v_i.$$

[2]Among these twisted sectors there is one isomorphic to $\mathcal{X}$ itself, which is sometimes called the *untwisted sector*.

or equivalently, the unique element in $\mathrm{Box}(\sigma(\gamma))$ that satisfies

$$\gamma^\vee = -\gamma \mod \sum_{i\in\sigma} \mathbb{Z}v_i.$$

2.2 Secondary fans and toric wall-crossing

The birational geometry of a toric variety (or more generally, toric Deligne-Mumford stacks) is described by a combinatorial object called *secondary fan*.

2.2.1 Secondary fan

We briefly recall the basic definitions and properties of secondary fan, following [15].

Let Σ be a triangulation of the cone C based on the set of vertices $\{v_1, v_2, \cdots, v_n\}$. We define the characteristic function $\varphi_\Sigma : \{v_1, v_2, \cdots, v_n\} \to \mathbb{R}$ by $\varphi_\Sigma(v_i) := \sum \mathrm{Vol}(\sigma)$, where $\mathrm{Vol}(\sigma)$ denotes the volume of σ, and the sum is taken over all simplexes σ in Σ that contain v_i as a vertex. Note that φ_Σ can be seen as a lattice point in $\mathbb{R}^n$.

Definition 13. The *secondary polytope* of the cone C is defined to be the convex hull of φ_Σ for all triangulations Σ in $\mathbb{R}^n$, and the *secondary fan* of C is defined to be the normal fan of the secondary polytope.

The following basic properties of secondary polytopes and fans can be found in [15, Chapter 7].

Proposition 14. The vertices of the secondary polytope of C (or equivalently, the maximal cones of the secondary fan of C) are in 1-1 correspondence with regular triangulations of C.

Remark 15. The intersection of two maximal cones of the secondary fan is a face in each. If it is of codimension 1, then the corresponding triangulations are said to be *adjacent* to each other.

2.2.2 Toric wall-crossings

Let Σ_- and Σ_+ be two adjacent triangulations of the cone C in the sense that the intersection of their corresponding maximal cones (which we denoted by C_{Σ_-} and C_{Σ_+} respectively) in the secondary fan is a codimension 1 cone. Then there exists a *circuit* (i.e., a minimal linearly dependent set) I defined by an integral linear relation

$$h_1 v_1 + \cdots + h_n v_n = 0$$

with $I = I_+ \sqcup I_-$ where $I_+ = \{i : h_i > 0\}$ and $I_- = \{i : h_i < 0\}$. Moreover, the linear relation $h = (h_1, \cdots, h_n)$ gives the defining equation of the codimension 1 subspace that is spanned by the intersection of the maximal cones corresponding to $\Sigma_\pm$. We specify a special class of cones in the fan $\Sigma_\pm$.

Definition 16. A maximal cone in $\Sigma_\pm$ of the form $\mathcal{F} \sqcup (I \backslash i)$ where $i \in I_\pm$ and $\mathcal{F} \subseteq \{1, 2, \cdots, n\} \backslash I$ is called an *essential maximal cone* in $\Sigma_\pm$, and the set $\mathcal{F}$ is called the *separating set* of the essential maximal cone. We denote the set of essential cones in $\Sigma_\pm$ by $\Sigma_\pm^{\text{es}}$. If the minimal cone $\sigma(\gamma_\pm)$ of a twisted sector $\gamma_\pm$ is a subcone of an essential maximal cone in Σ, then we say $\gamma_\pm$ is an *essential twisted sector*, and denote the set of essential twisted sectors by $\text{Box}(\Sigma_\pm^{\text{es}})$.

Definition 17. Let $\sigma_\pm$ be essential maximal cones in $\Sigma_\pm$. We say that σ_+ and σ_- are *adjacent* if they have the same separating set $\mathcal{F}$. Equivalently, σ_- can be obtained from σ_+ by adding the vector $i \in I_+$ which is missing in σ_+ and deleting some vector $k \in I_-$.

It is proved in [15, §7.1] that one can obtain one triangulation of $\Sigma_\pm$ from another by replacing all essential cones of one triangulation with those of another. The associated toric Deligne-Mumford stacks $\mathbb{P}_{\Sigma_\pm}$ are then related by an Atiyah flop that is a composition of a weighted blow-down and a weighted blow-up:

$$\begin{array}{ccc} & \mathbb{P}_{\hat{\Sigma}} & \\ {}^{f_-}\swarrow & & \searrow^{f_+} \\ \mathbb{P}_{\Sigma_-} & \dashrightarrow & \mathbb{P}_{\Sigma_+} \end{array}$$

Here $\mathbb{P}_{\hat{\Sigma}}$ is a common blow-up of $\mathbb{P}_{\Sigma_\pm}$ defined as follows. The linear relation can be rewritten as:

$$\sum_{i \in I_+} h_i v_i = -\sum_{i \in I_-} h_i v_i$$

We denote this vector by $\hat{v}$. We then define $\hat{\Sigma}$ to be the fan obtained by replacing all essential cones of $\Sigma_\pm$ by cones of the form $\mathcal{F} \cup \{\hat{v}\} \cup (I \backslash \{i_+, i_-\})$ where $i_\pm \in I_\pm$.

2.2.3 Behavior of twisted sectors under toric wall-crossings

The behavior of twisted sectors under the wall-crossing was studied in [10, §4] and [13, §6.2.3]. In this subsection, we prove a technical lemma that will be used in Section 3.1.

From now on we will use the symbol γ to denote either a connected component of the inertia stack or its corresponding lattice points in $C \cap N$. Now we give an alternative characterization of twisted sectors following [21]. For any lattice point $c \in C \cap N$ we define

$$\begin{aligned} \mathbb{K}_c :&= \left\{ (l_i) \in \mathbb{Q}^n : \ \sum_{i=1}^{n} l_i v_i = -c, \ \{i : l_i \notin \mathbb{Z}\} \text{ is a cone in } \Sigma \right\} \\ &= \bigcup_{\gamma \in \mathrm{Box}(\Sigma)} L_{c,\gamma} \end{aligned}$$

where

$$L_{c,\gamma} := \left\{ (l_i) \in \mathbb{Q}^n : \sum_{i=1}^{n} l_i v_i = -c, \ l_i \equiv \gamma_i \mod \mathbb{Z} \right\}$$

Clearly the set $L := L_{0,0}$ acts on $\mathbb{K}_c$ by translation. The following characterization of twisted sectors can be found in [21, §3.1.3].

Lemma 18. There is an injection $\mathbb{K}_c/L \hookrightarrow \mathrm{Box}(\Sigma)$, and the image of this map consists of twisted sectors γ such that the set $L_{c,\gamma}$ is non-empty.

Proof. The map $\mathbb{K}_c \to \mathrm{Box}(\Sigma)$ defined by $(l_i) \mapsto \sum_{i=1}^{n} \{l_i\} v_i$ clearly factors through $\mathbb{K}_c/L$. Now take a twisted sector $\gamma \in \mathrm{Box}(\Sigma)$, then any element in the lattice $L_{c,\gamma}$ is mapped to γ due to the condition $l_i \equiv \gamma_i \mod \mathbb{Z}$. □

Thus each twisted sector γ with $L_{c,\gamma} \neq \emptyset$ is represented by elements in the lattice $\mathbb{K}_c$. We call such representatives the *liftings* of γ.

Definition 19. Let $\gamma_\pm \in \mathrm{Box}(\Sigma_\pm)$ be two essential twisted sectors. We say that γ_- is *adjacent* to γ_+ if there exists a pair of essential maximal cones $\sigma_\pm$ in $\Sigma_\pm$ such that $\sigma(\gamma_+)$ and $\sigma(\gamma_-)$ are subcones of σ_+ and σ_- respectively.

Lemma 20. Let $\Sigma_\pm$ be two adjacent triangulations. Then there exists a choice of the lifting $\mathrm{Box}(\Sigma_\pm) \to \mathbb{K}_c^\pm$ such that any pair of adjacent twisted sectors γ_+ and γ_-, the lifting $\tilde{\gamma_+}$ and $\tilde{\gamma_-}$ differs by a rational multiple of the defining linear relation $h = (h_1, \cdots, h_n)$ of the circuit I that corresponds to the wall-crossing $\Sigma_+ \to \Sigma_-$.

Proof. The proof is similar to [10, Proposition 4.4(ii)]. We begin with an arbitrary essential twisted sector $\gamma_+ \in \mathrm{Box}(\Sigma_+^{\mathrm{es}})$ and an arbitrary lifting $\gamma_+ = \sum_{j=1}^{n} (\gamma_+)_j v_j$. We write $\sigma(\gamma_+) \subseteq \mathcal{F} \sqcup I \backslash i$ where $\mathcal{F}$ is a separating set and $i \in I_+$. For any $k \in I_-$ we take a rational number $q \in \mathbb{Q}$ such that $(\gamma_+)_k + q h_k \in \mathbb{Z}$. Then $((\gamma_+)_j)$ and $((\gamma_+)_j + q h_j)$ differ by a rational multiple of h. We denote the associated twisted sector of the latter by γ_-. It's then clear that $\sigma(\gamma_-) \subseteq \mathcal{F} \sqcup I \backslash k$, hence γ_- is an essential twisted sector of Σ_-. Moreover, since $\sigma(\gamma_\pm)$ share the same separating set, σ_+ and σ_- are adjacent.

Now we have proved that by adding an appropriate rational multiple of h to a lifting of an essential twisted sector in Σ_+, we get a lifting of an essential twisted sector in Σ_-. It remains to show that any essential twisted sector in Σ_- can be obtained in this way. To see this, note that the procedure above is invertible (i.e., adding $-q \cdot h$ to the lifting), so if we start with some γ_+, apply the procedure above from Σ_+ to Σ_- and back, we recover the original twisted sector. By switching the roles of Σ_+ and Σ_-, this shows that any essential twisted sector in Σ_- can be obtained from this procedure. □

Remark 21. In the proof above we see that for a fixed essential twisted sector γ_+ of Σ_+ and $k \in I_-$, the lifting we constructed for the adjacent twisted sector is not unique due to the freedom of the condition $(\gamma_+)_k + qh_k \in \mathbb{Z}$. However, it is important to note that any two such liftings differ by an integral multiple of h. In fact, whenever we have two liftings $\gamma_+ + q_1 h$ and $\gamma_+ + q_2 h$, they both define the same twisted sector of Σ_- if and only if the fractional part of the corresponding coordinates are equal. This means that $(q_1 - q_2)h$ should have integral coordinates. The primitivity of h then forces $q_1 - q_2$ to be an integer.

2.3 Derived categories, K-theory and orbifold cohomology

In this section, we review basic facts about derived categories, K-theories and orbifold cohomology of smooth toric Deligne-Mumford stacks, and fix the notations that will be used throughout this paper. The main references are [5, 8, 9].

2.3.1 Derived categories of toric stacks

Let $\mathbb{P}_{\mathbf{\Sigma}}$ be the toric stack associated to the stacky fan $\mathbf{\Sigma}$. We denote the usual bounded derived category of coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ by $D^b(\mathbb{P}_{\mathbf{\Sigma}})$. Such derived categories have been studied extensively in literature, see for example [24].

Since the toric stacks we studied in this dissertation are generally only semi-projective and hence not necessarily compact, we need another version of derived categories, the *compactly supported derived category*, which we denote by $D^b_c(\mathbb{P}_{\mathbf{\Sigma}})$.

Definition 22. Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a smooth toric Deligne-Mumford stack cooresponding to a stacky fan $\mathbf{\Sigma}$. We denote the admissible subcategory of $D^b(\mathbb{P}_{\mathbf{\Sigma}})$ that consists of complexes whose cohomology are supported on the compact toric divisors of $\mathbb{P}_{\mathbf{\Sigma}}$ by $D^b_c(\mathbb{P}_{\mathbf{\Sigma}})$.

2.3.2 K-theory of toric stacks

In this subsection we look at the K-groups of the toric stack $\mathbb{P}_{\mathbf{\Sigma}}$. Again, there are two types of K-groups, the ordinary K-group $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ and the compactly-supported K-group $K_0^c(\mathbb{P}_{\mathbf{\Sigma}})$, which are defined as the Grothendieck groups of the derived categories $D^b(\mathbb{P}_{\mathbf{\Sigma}})$ and $D^b_c(\mathbb{P}_{\mathbf{\Sigma}})$ introduced in the previous subsection. We have the following combinatorial descriptions.

Proposition 23. Let C, v_i and Σ be as before. We denote the class of the line bundle $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}(D_i)$ corresponding to the ray v_i by R_i. Then $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ is isomorphic to the quotient of the ring $\mathbb{C}[R_i^{\pm 1}]$ by the relations

$$\prod_{i=1}^{n} R_i^{\mu(v_i)} - 1, \quad \mu \in N^\vee, \text{ and } \prod_{i\in I}(1-R_i), \quad I \notin \Sigma$$

Furthermore, if we denote the class of the structure sheaf of the closed substack corresponding to a cone σ_I by G_I, then $K_0^c(\mathbb{P}_{\mathbf{\Sigma}})$ is a module over $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ generated by G_I for all $I \in \Sigma$ with σ_I being an interior cone, with the relations given by

$$(1 - R_i^{-1})G_I = G_{I\cup\{i\}} \text{ if } I \cup \{i\} \in \Sigma \text{ and } 0 \text{ otherwise}, \ \forall i.$$

Proof. See [8, Proposition 3.3, Definition 3.9]. □

There is a natural non-degenerate pairing $\chi(-,-)$ between $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ and $K_0^c(\mathbb{P}_{\mathbf{\Sigma}})$ called *Euler characteristic pairing* defined as the alternative sum of the dimension of Ext groups. More precisely, let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes in the derived categories $D^b(\mathbb{P}_{\mathbf{\Sigma}})$ and $D^b_c(\mathbb{P}_{\mathbf{\Sigma}})$ respectively, we define

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \sum_{i=0}^{\infty} \dim \operatorname{Hom}_{D^b(\mathbb{P}_{\mathbf{\Sigma}})}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]).$$

In particular, if we take $\mathcal{F}^\bullet$ to be the structure sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$ and $\mathcal{G}^\bullet$ to be a coherent sheaf, then this definition recovers the usual Euler characteristic of coherent sheaves.

2.3.3 Orbifold cohomology of toric stacks

In this subsection we descend further from the K-theories to the orbifold cohomology groups. Again there are two types of orbifold cohomology theory associated to it, namely the usual orbifold cohomology and the orbifold cohomology with compact support. They are defined as the direct sum[3] of the usual cohomology spaces of the twisted sectors.

We have the following combinatorial description of the cohomology spaces of twisted sectors.

Proposition 24. As usual, $\operatorname{Star}(\sigma(\gamma))$ denotes the set of cones in Σ that contain $\sigma(\gamma)$. Cohomology space H_γ of the twisted sector γ is naturally isomorphic to the

[3] Strictly speaking the degrees of direct summands need to be modified by *ages* of the corresponding twisted sectors. We omit them here since they will play no role in the main results.

quotient of the polynomial ring $\mathbb{C}[D_i : i \in \mathrm{Star}(\sigma(\gamma))\backslash\sigma(\gamma)]$ by the ideal generated by the relations

$$\prod_{j\in J} D_j,\ J \notin \mathrm{Star}(\sigma(\gamma)), \quad \text{and} \quad \sum_{i\in \mathrm{Star}(\sigma(\gamma))\backslash\sigma(\gamma)} \mu(v_i)D_i,\ \mu \in \mathrm{Ann}(v_i, i \in \sigma(\gamma)).$$

There is a $\mathbb{C}[D_1, \ldots, D_n]$-module structure on H_γ defined by declaring $D_i = 0$ for $i \notin \mathrm{Star}(\sigma(\gamma))$ and solving (uniquely) for $D_i, i \in \sigma(\gamma)$ to satisfy the linear relations $\sum_{i=1}^n \mu(v_i)D_i = 0$ for all $\mu \in N^\vee$.

Moreover, the cohomology space with compact support H_γ^c is generated by F_I for $I \in \mathrm{Star}(\sigma(\gamma))$ such that $\sigma_I^\circ \subseteq C^\circ$ with relations

$$\begin{gathered} D_iF_I - F_{I\cup\{i\}} \text{ for } i \notin I, I \cup \{i\} \in \mathrm{Star}(\sigma(\gamma)) \\ \text{and } D_iF_I \text{ for } i \notin I, I \cup \{i\} \notin \mathrm{Star}(\sigma(\gamma)) \end{gathered}$$

as a module over H_γ.

There is a natural integration map $\int$ defined on each cohomology space H_γ^c with compact support.

Proposition 25. There exists a unique linear function $\int_\gamma : H_\gamma^c \to \mathbb{C}$ that takes values $\frac{1}{\mathrm{Vol}_{\overline{I}}}$ on each generator F_I with $|I| = d+1-|\sigma(\gamma)|$ (i.e., of highest degree), where $\mathrm{Vol}_{\overline{I}}$ denotes the volume of the cone $\overline{\sigma_I}$ in the quotient fan $\Sigma/\sigma(\gamma)$. Moreover, it takes value zero on all elements of H_γ^c of lower degree.

Proof. See [8, Proposition 2.6]. □

Now we give the definition of orbifold cohomology of a toric stack $\mathbb{P}_{\mathbf{\Sigma}}$.

Definition 26. The orbifold cohomology $H^*_{\mathrm{orb}}(\mathbb{P}_{\mathbf{\Sigma}})$ of the smooth toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ is defined as the direct sum $\bigoplus_\gamma H_\gamma$ over all twisted sectors. Similarly, the orbifold cohomology with compact support $H^*_{\mathrm{orb},c}(\mathbb{P}_{\mathbf{\Sigma}})$ is defined as $\bigoplus_\gamma H_\gamma^c$. We denote by 1_γ the generator of H_γ.

Remark 27. There is an involution map $*$ on the orbifold cohomology $H^*_{\text{orb}}(\mathbb{P}_{\mathbf{\Sigma}})$ that maps H_γ to $H_{\gamma^\vee}$, defined by $(1_\gamma)^* = 1_{\gamma^\vee}$ and $(D_i)^* = -D_i$.

Remark 28. One can define an integration map $\int : H^*_{\text{orb},c}(\mathbb{P}_{\mathbf{\Sigma}}) \to \mathbb{C}$ by taking direct sum of the integration map on each twisted sectors.

The K-groups $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ and $K_0^c(\mathbb{P}_{\mathbf{\Sigma}})$ are related to the orbifold cohomology $H^*_{\text{orb}}(\mathbb{P}_{\mathbf{\Sigma}})$ and $H^*_{\text{orb},c}(\mathbb{P}_{\mathbf{\Sigma}})$ by the following combinatorial Chern characters.

Proposition 29. There is a natural isomorphism

$$ch : K_0(\mathbb{P}_{\mathbf{\Sigma}}) \xrightarrow{\sim} H^*_{\text{orb}}(\mathbb{P}_{\mathbf{\Sigma}}) = \bigoplus_{\gamma \in \text{Box}(\Sigma)} H_\gamma$$

defined by

$$ch_\gamma(R_i) = \begin{cases} 1, & i \notin \text{Star}(\sigma(\gamma)) \\ e^{D_i}, & i \in \text{Star}(\sigma(\gamma)) \backslash \sigma(\gamma) \\ e^{2\pi \mathrm{i} \gamma_i} \prod_{j \notin \sigma(\gamma)} ch_\gamma(R_j)^{\mu_i(v_j)}, & i \in \sigma(\gamma) \end{cases}$$

Similarly, there is a natural isomorphism

$$ch^c : K_0^c(\mathbb{P}_{\mathbf{\Sigma}}) \xrightarrow{\sim} H^*_{\text{orb},c}(\mathbb{P}_{\mathbf{\Sigma}}) = \bigoplus_{\gamma \in \text{Box}(\Sigma)} H^c_\gamma$$

defined by

$$ch^c_\gamma(\prod_{i=1}^n R_i^{l_i} G_I) = \begin{cases} 0, & I \not\subseteq \text{Star}(\sigma(\gamma)) \\ \prod_{i=1}^n ch_\gamma(R_i^{l_i}) F_I, & I \subseteq \text{Star}(\sigma(\gamma)) \end{cases}$$

Proof. See [8, Proposition 3.7, 3.11]. □

Via the combinatorial Chern characters, the Euler pairing defined on K-theories can be translated to the orbifold cohomology spaces $H^*_{\mathrm{orb}}(\mathbb{P}_{\boldsymbol{\Sigma}})$ and $H^*_{\mathrm{orb},c}(\mathbb{P}_{\boldsymbol{\Sigma}})$. In terms of orbifold cohomology, the pairing has the following explicit formula.

Proposition 30. The Euler characteristic pairing $\chi : H^*_{\mathrm{orb}}(\mathbb{P}_{\boldsymbol{\Sigma}}) \otimes H^*_{\mathrm{orb},c}(\mathbb{P}_{\boldsymbol{\Sigma}}) \to \mathbb{C}$ on the toric DM stack $\mathbb{P}_\Sigma$ is given by

$$\chi(a,b) = \chi(\oplus_\gamma a_\gamma, \oplus_\gamma b_\gamma) = \sum_\gamma \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_{\gamma^\vee} \operatorname{Td}(\gamma^\vee) a^*_\gamma b_{\gamma^\vee}$$

Here $\operatorname{Td}(\gamma)$ is the Todd class of the twisted sector γ, defined as

$$\operatorname{Td}(\gamma) = \frac{\prod_{i \in \operatorname{Star}\sigma(\gamma) \setminus \sigma(\gamma)} D_i}{\prod_{i \in \operatorname{Star}\sigma(\gamma)} (1 - e^{-D_i})}.$$

Proof. See [6, Lemma 4.20]. □

The following easy consequence will be used in Section 4.3.

Corollary 31. The Euler characteristic of the sheaves represented by the class $v \in K^c_0(\mathbb{P}_{\boldsymbol{\Sigma}})$ is given by

$$\chi(v) = \sum_{\gamma \in \operatorname{Box}(\Sigma)} \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_\gamma ch^c_\gamma(v) \operatorname{Td}(\gamma).$$

Finally, there is a special type of characteristic classes of the smooth toric DM stack $\mathbb{P}_{\boldsymbol{\Sigma}}$, called *Gamma classes*, which play an essential role in the computation of this dissertation. A similar definition has been introduced in [21] for general smooth DM stacks. The version we used in this paper comes from [5, Corollary 3.14].

Definition 32. For each twisted sector γ of $\mathbb{P}_{\boldsymbol{\Sigma}}$, we define its *Gamma class* by

$$\widehat{\Gamma}_\gamma = \prod_{i \in \sigma(\gamma)} \Gamma(\gamma_i + \frac{D_i}{2\pi \mathrm{i}}) \prod_{i \in \operatorname{Star}(\sigma(\gamma)) \setminus \sigma(\gamma)} \Gamma(1 + \frac{D_i}{2\pi \mathrm{i}})$$

which is a cohomology class in H^*_γ. We define the *Gamma class* of $\mathbb{P}_{\mathbf{\Sigma}}$ to be the direct sum of Gamma classes of all of its twisted sectors.

2.4 Better-behaved GKZ hypergeometric systems

In this section, we introduce the main object studied in this dissertation, the *better-behaved GKZ hypergeometric systems*, and recall some basic properties and known results.

2.4.1 Basic definitions

We recall the combinatorial setting from Chapter 1. Let C be a finite rational polyhedral cone in a lattice $N = \mathbb{Z}^{\mathrm{rk} N}$. We assume that all ray generators of C lie on a primitive hyperplane $\deg(\cdot) = 1$ where $\deg : N \to \mathbb{Z}$ is a linear function. This data encodes an affine toric variety $X = \operatorname{Spec} \mathbb{C}[N^\vee \cap C^\vee]$, with the hyperplane condition equivalent to X being Gorenstein, i.e. having trivial dualizing sheaf.

Let $\{v_i\}_{i=1}^n$ be a set of n lattice points in C which includes all of its ray generators, with $\deg(v_i) = 1$ for all i. One can construct (stacky) crepant resolutions $\mathbb{P}_{\mathbf{\Sigma}} \to X$, where the stacky fan $\mathbf{\Sigma}$ is obtained by subdivisions Σ of C based on triangulations that involve some of the points v_i. Note that the additional data $\{v_i\}$ in the definition of $\mathbf{\Sigma}$ is chosen to be these deg 1 points.

As we have already explained in the introduction, a particular case of Kawamata-Orlov $K \to D$ conjecture (a.k.a. DK conjecture) asserts that the derived categories of coherent sheaves on $\mathbb{P}_\Sigma$ are independent of the choice of Σ. In fact, it is expected that there is an isotrivial family of triangulated categories which interpolates between the categories in question. This rather mysterious family is well understood at the level of complexified Grothendieck K-groups. Namely, these should correspond to solutions of a certain version of the Gel'fand-Kapranov-Zelevinsky system of hypergeometric

PDEs. In fact, due to non-compactness of X and $\mathbb{P}_\Sigma$, there are two such systems, denoted by bbGKZ$(C,0)$ and bbGKZ$(C^\circ,0)$, conjecturally dual to each other [8]. In the appropriate limit that corresponds to the triangulation Σ, solutions to these systems can be identified with usual and compactly supported orbifold cohomology of $\mathbb{P}_\Sigma$ by means of two special Gamma series.

Now we give the definition of better-behaved GKZ systems.

Definition 33. Consider the system of partial differential equations on the collection of functions $\{\Phi_c(x_1,\ldots,x_n)\}$ in complex variables $x_1,\ldots,x_n$, indexed by the lattice points in C:

$$\partial_i\Phi_c = \Phi_{c+v_i}, \quad \sum_{i=1}^n \langle\mu, v_i\rangle x_i\partial_i\Phi_c + \langle\mu, c\rangle\Phi_c = 0$$

for all $\mu\in N^\vee$, $c\in C$ and $i=1,\ldots,n$. We denote this system by bbGKZ$(C,0)$. Similarly by considering lattice points in the interior C° only, we can define bbGKZ$(C^\circ,0)$.

Remark 34. A similar definition was introduced by Hiroshi Iritani [22] (who used the term *multi-GKZ systems*) in his study of the quantum D-module associated to a toric complete intersection and the periods of its mirror.

2.4.2 Gamma series solutions

Basic properties of such systems have been established in a series of papers of Borisov and Horja. We briefly recall some of them.

The following result expresses the local solutions of the bbGKZ systems near a large radius limit point corresponding to a toric stack $\mathbb{P}_{\mathbf{\Sigma}}$ as a Gamma series which takes values in the orbifold cohomology (or equivalently, the K-theories) of the corresponding toric stack, thus explains the reason why such systems are naturally decategorification of the conjectural isotrivial family of derived categories.

Proposition 35. Fix a triangulation Σ of the cone C. We define the cohomology-valued Gamma series as

$$\Gamma_c = \bigoplus_{\gamma \in \mathrm{Box}(\Sigma)} \sum_{l \in L_{c,\gamma}} \prod_{j=1}^{n} \frac{x_j^{l_j + \frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1 + l_j + \frac{D_j}{2\pi \mathrm{i}})}$$

for each lattice point $c \in C$. Then for any linear function f on the orbifold cohomology $H^*_{orb}(\mathbb{P}_{\mathbf{\Sigma}})$, the composition $f \circ \Gamma_c$ converges in the following region

$$U_\Sigma = \{(x_j) \in \mathbb{C}^n : (-\log|x_j|) \in \hat{\psi} + C_\Sigma,\ \arg(\mathbf{x}) \in (-\pi, \pi)^n\}$$

where $\hat{\psi}$ is a point in the maximal cone C_Σ of the secondary fan corresponding to Σ. This should be thought of as a neighborhood of the large radius limit point corresponding to Σ.

A similar construction exists for the dual system bbGKZ$(C^\circ, 0)$.

Proposition 36. We define the Gamma series which takes values in the compactly supported orbifold cohomology as

$$\Gamma_c^\circ(x_1, \ldots, x_n) = \bigoplus_{\gamma} \sum_{l \in L_{c,\gamma}} \prod_{i=1}^{n} \frac{x_i^{l_i + \frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1 + l_i + \frac{D_i}{2\pi \mathrm{i}})} \left(\prod_{i \in \sigma} D_i^{-1} \right) F_\sigma$$

where σ is the set of i with $l_i \in \mathbb{Z}_{<0}$ and F_σ's are generators of $H^{*,c}_{\mathrm{orb}}$ as a module over H^*_{orb}. Then for any linear function f on the compactly supported orbifold cohomology $H^*_{orb,c}(\mathbb{P}_{\mathbf{\Sigma}})$, the composition $f \circ \Gamma_c^\circ$ converges in the following region

$$U_\Sigma = \{(x_j) \in \mathbb{C}^n : (-\log|x_j|) \in \hat{\psi} + C_\Sigma,\ \arg(\mathbf{x}) \in (-\pi, \pi)^n\}$$

where $\hat{\psi}$ is a point in the maximal cone C_Σ of the secondary fan corresponding to Σ.

This system gives a holonomic system of PDEs. It follows from the general theory of holonomic D-modules that its rank (i.e., the dimension of the solution space) is finite. For more background on this, we refer to [20]. In contrast to the usual GKZ system where rank jumps may occur at non-generic parameters (see [26]), it is proved in [7] that the better-behaved GKZ systems always have the expected rank which is equal to the normalized volume of the convex hull of ray generators of the cone C.

2.4.3 Duality of bbGKZ systems

It has been previously conjectured in [8] that the systems bbGKZ$(C,0)$ and bbGKZ$(C^\circ,0)$ are dual to each other, in the sense that there is a pairing $\langle\cdot,\cdot\rangle$ between solutions $\Phi=(\Phi_c)$ and $\Psi=(\Psi_d)$ thereof in the form

$$\langle \Phi, \Psi\rangle = \sum_{c,d} p_{c,d}(\mathbf{x})\Phi_c\Psi_d,$$

where $p_{c,d}$ are polynomials in $\mathbf{x}$, with only finitely many of them nonzero. This pairing should be constant in $\mathbf{x}$ and could be viewed as the duality of the local systems of solutions. A nontrivial example of this duality has been verified in [8] and the $\mathrm{rk}(N)=2$ case has been settled affirmatively in [6]. Moreover, in certain regions of $\mathbf{x}$ that roughly correspond to the complexified Kähler cones of $\mathbb{P}_\Sigma$, one can construct solutions of bbGKZ$(C,0)$ and bbGKZ$(C^\circ,0)$ with values in certain cohomology or K-theory groups of $\mathbb{P}_\Sigma$. Then it was conjectured in [8] that the above pairing should give (up to a constant) the inverse of a certain Euler characteristic pairing on these spaces.

In [5], Borisov and the author were able to verify both statements and thus prove Conjecture 7.3 of [8] in full generality. In fact, we proved a slightly stronger statement which allows for arbitrary parameter $\beta\in\mathbb{C}^n$. We briefly recall the main result here.

Specifically, the following formula provides the pairing in question. Let $v\in C^\circ$ be an element in general position. For a subset $I\subseteq\{1,\ldots,n\}$ of size $\mathrm{rk}N$ we consider the cone $\sigma_I = \sum_{i\in I}\mathbb{R}_{\geq 0}v_i$. We define the coefficients $\xi_{c,d,I}$ for $c+d=v_I$ as

$$\xi_{c,d,I} = \begin{cases} (-1)^{\deg(c)}, \text{ if } \dim\sigma_I = \operatorname{rk} N \text{ and both } c+\varepsilon v \text{ and } d-\varepsilon v \in \sigma_I^\circ \\ 0, \text{ otherwise.} \end{cases}$$

Here the condition has to hold for all sufficiently small $\varepsilon > 0$. As usual, we denote by Vol_I the absolute value of the determinant of the matrix of coefficients of v_i, $i \in I$ in a basis of N (i.e., the normalized volume of I).

Theorem 37. *For any pair of solutions* (Φ_c) *and* (Ψ_d) *of* $\mathrm{bbGKZ}(C,\beta)$ *and* $\mathrm{bbGKZ}(C^\circ,-\beta)$ *respectively, the pairing*

$$\langle\Phi,\Psi\rangle = \sum_{c,d,I}\xi_{c,d,I}\,\mathrm{Vol}_I\left(\prod_{i\in I}x_i\right)\Phi_c\Psi_d$$

is a constant.

Now we restrict to the case $\beta = 0$ since this is the case related to mirror symmetry. As was mentioned in the last section, for a regular triangulation Σ there is a description of solutions to $\mathrm{bbGKZ}(C,0)$ and $\mathrm{bbGKZ}(C^\circ,0)$ in terms of the Gamma series $\Gamma = (\Gamma_c)$ and $\Gamma^\circ = (\Gamma_d^\circ)$ with values in the (usual and compactly supported) orbifold cohomology $H^*_{orb}(\mathbb{P}_{\mathbf{\Sigma}})$ and $H^*_{orb,c}(\mathbb{P}_{\mathbf{\Sigma}})$ associated to $\mathbb{P}_\Sigma$.

Theorem 38. *The constant pairing* $\langle\Gamma,\Gamma^\circ\rangle$ *is equal up to a constant factor to the inverse of the Euler characteristic pairing* $\chi(-,-): H^*_{orb}(\mathbb{P}_{\mathbf{\Sigma}})\otimes H^*_{orb,c}(\mathbb{P}_{\mathbf{\Sigma}})\to\mathbb{C}$.

2.5 A D-module formulation of bbGKZ systems

The (original version of) GKZ hypergeometric systems have been extensively studied by using the theory of D-modules, see e.g. [26, 28]. The better-behaved GKZ systems can also be formulated in terms of D-modules. Since the content of this section will not be used in the remainder of this dissertation, we provide only a very brief introduction. For a comprehensive study of the theory of D-modules, we refer the readers to [20].

We denote by D the *Weyl algebra with n variables* defined by

$$D = \mathbb{C}[x_1, \cdots, x_n]\langle \partial_1, \cdots, \partial_n \rangle$$

where there are relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$ for any i, j. Note that every element in D can be written as a finite sum

$$\sum_{i_1, \cdots, i_n} f_{i_1, \cdots, i_n}(x_1, \cdots, x_n) \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$$

More generally one can consider the *sheaf of differential operators* on an arbitrary algebraic variety X. We focus on the affine case $X = \mathbb{C}^n$ in this dissertation.

There is a close relation between left modules over D with systems of linear PDEs. As an simple example, let $P \in D$ be a differential operator, and consider the differential equation $Pu = 0$, where $u \in \mathcal{O}_{hol}$ is a holomorphic function on $\mathbb{C}^n$. We associate a left D-module $M = D/DP$ to this equation. It is easy to see that the space of classical solutions to this equation is naturally given by $\mathrm{Hom}_D(M, \mathcal{O}_{hol})$.

There is a particularly nice class of D-modules, called *holonomic D-modules*, whose precise definition is given in [20, Section 3.1]. Roughly speaking, holonomic modules corresponds to systems of linear PDEs whose solution spaces are finite-dimensional.

There exists a duality functor on the derived categories of coherent D-modules

$$\mathbb{D} : D^b_{coh}(D_X - mod) \to D^b_{coh}(D_X - mod)^{op}$$

defined as

$$M^\bullet \mapsto (R\mathcal{H}om_{D_X}(M^\bullet, D_X) \otimes_{\mathcal{O}_X} \omega_X^\vee)\,[\dim X]$$

where ω_X is the dualizing sheaf of X. The dual of a holonomic D-module is also holonomic.

Definition 39. The *bbGKZ D-module* $\mathrm{bbGKZ}(C,\beta)$ associated to the cone C and parameter β is defined as the left D-module

$$\bigoplus_{c\in C\cap N} D\cdot 1_c \Big/ \left(\sum_{c\in C\cap N, i=1,\cdots,n} D\cdot(\partial_j 1_c - 1_{c+v_i}) + \sum_{c\in C\cap N, \mu\in N^\vee} D\cdot (E_\mu - \mu(\beta - c))\cdot 1_c \right)$$

where $E_\mu = \sum_{i=1}^n \mu(v_i)x_i\partial_i$ is the Euler operator. We can define $\mathrm{bbGKZ}(C^\vee,\beta)$ in a similar manner by replacing C by C°.

Reichelt, Sevenheck and Walther [29] have studied bbGKZ systems in terms of D-modules and Euler-Koszul complexes. In particular, they proved that the systems $\mathrm{bbGKZ}(C,\beta)$ and $\mathrm{bbGKZ}(C^\circ,-\beta)$ are holonomic dual to each other, up to a shift of grading. The holonomic duality then implies the existence of a non-degenerate pairing between these two D-modules (for the precise statement, see [29, Theorem 5.9]). While their approach is more general and a priori contains more information than the pairing introduced by Borisov and the author, the explicit formula of the latter allows one to compute the asymptotics towards the large radius limits and to verify the GKZ pairing actually recovers the Euler pairing on the associated toric stacks. In principle, their pairing should agree with the explicit formula we provided here, however by the time this dissertation was finished we have not been able to prove it.

Chapter 3

Analytic continuation of Gamma series and Fourier-Mukai transforms

In this chapter we prove the first main result of this dissertation:

Theorem 40 (= Theorem 54, 55). *The following diagrams commute:*

$$\begin{array}{ccc} K_0(\mathbb{P}_{\mathbf{\Sigma}_+})^\vee & \xrightarrow{-\circ\Gamma_+} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_+)) \\ \downarrow{\scriptstyle \mathrm{FM}^\vee} & & \downarrow{\scriptstyle \mathrm{MB}} \\ K_0(\mathbb{P}_{\mathbf{\Sigma}_-})^\vee & \xrightarrow{-\circ\Gamma_-} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_-)) \end{array}$$

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\mathbf{\Sigma}_+})^\vee & \xrightarrow{-\circ\Gamma_+^\circ} & Sol(\mathrm{bbGKZ}(C^\circ),U_+) \\ \downarrow{\scriptstyle (\mathrm{FM}^c)^\vee} & & \downarrow{\scriptstyle \mathrm{MB}^c} \\ K_0^c(\mathbb{P}_{\mathbf{\Sigma}_-})^\vee & \xrightarrow{-\circ\Gamma_-^\circ} & Sol(\mathrm{bbGKZ}(C^\circ),U_-) \end{array}$$

where the horizontal arrows are mirror symmetry maps, FM *(*FM^c*) and* MB *(*MB^c*) denote the Fourier-Mukai transforms and analytic continuation transformations of* *solutions respectively.*

This chapter is organized as follows. In Section 3.1 we compute the analytic continuation of Gamma series solution by applying the Mellin-Barnes integral method. In Section 3.2 we give a combinatorial formula for the pullback-pushforward functor associated to the toric wall-crossing, and observe that it coincides with the computation of the previous section. In Section 3.3 we utilize the duality result on the bbGKZ systems to deduce the parallel results for the dual systems.

3.1 Analytic continuation of Gamma series

In this section we compute the analytic continuation of Gamma series solutions to bbGKZ$(C, 0)$.

Recall from Section 2.4.2 that the Gamma series solution to bbGKZ$(C, 0)$ associated to a triangulation Σ is given by

$$\Gamma_c = \bigoplus_{\gamma\in \mathrm{Box}(\Sigma)} \sum_{l\in L_{c,\gamma}} \prod_{j=1}^{n} \frac{x_j^{l_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi \mathrm{i}})}$$

and there exists a point $\hat{\psi}$ in the maximal cone C_Σ of the secondary fan corresponding to Σ such that the series converges absolutely on the open set

$$U_\Sigma = \{(x_j)\in \mathbb{C}^n : (-\log|x_j|)\in \hat{\psi}+C_\Sigma,\ \arg(\mathbf{x})\in(-\pi,\pi)^n\}$$

which should be thought of as a neighborhood of the large radius limit point corresponding to Σ.

We introduce some additional notations that will be used later. We define $L_{c,\gamma,\sigma}$ to be the subset of $L_{c,\gamma}$ with the additional property that the following set

$$I(l) := \{i : l_i \in \mathbb{Z}_{<0}\} \sqcup \sigma(\gamma) = \{i : l_i \notin \mathbb{Z}_{\geq 0}\}$$

is a subcone of the maximal cone σ. Note that an element $l \in L_{c,\gamma}$ has a nonzero contribution to the series if and only if it lies in one of the subsets $L_{c,\gamma,\sigma}$. Along the same line as the proof of [5, Proposition 3.8] we can prove the following result. See also [10, Proposition 2.8] for a similar result for the usual GKZ systems.

Proposition 41. For each maximal cone σ, the subseries

$$\bigoplus_{\gamma\in\mathrm{Box}(\Sigma)} \sum_{l\in L_{c,\gamma,\sigma}} \prod_{j=1}^{n} \frac{x_j^{l_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi \mathrm{i}})}$$

is absolutely and uniformly convergent on compacts in the region

$$U_\sigma = \{(x_j) \in \mathbb{C}^n : (-\log|x_j|) \in \hat{\psi}_\sigma + C_\sigma,\ \arg(\mathbf{x}) \in (-\pi,\pi)^n\}$$

where C_σ denotes the union of all maximal cones C_Σ in the secondary fan that corresponds to triangulations Σ such that $\sigma \in \Sigma$, and $\hat{\psi}_\sigma$ is a point in C_σ.

Remark 42. More generally, for any subset of maximal cones J of Σ, the subseries taken over the union of all $L_{c,\gamma,\sigma}$ for $\sigma \in J$ converges absolutely and uniformly on compacts in $U_J := \cap_{\sigma\in J} U_\sigma$. In particular, the open set U_Σ is a subset of the intersection of U_σ's for all $\sigma \in \Sigma$.

Furthermore we define $L^{\mathrm{es}}_{c,\gamma}$ to be the union of $L_{c,\gamma,\sigma}$ for all $\sigma \in \Sigma^{\mathrm{es}}$. Note that $L^{\mathrm{es}}_{c,\gamma}$ is non-empty only if $\gamma \in \mathrm{Box}(\Sigma^{\mathrm{es}})$. We define the *essential part* $\Gamma^{\mathrm{es}}_c = \oplus_\gamma \Gamma^{\mathrm{es}}_{c,\gamma}$ of the Gamma series Γ_c to be the subseries that consists of terms corresponding to $l \in L^{\mathrm{es}}_{c,\gamma}$, namely

$$\Gamma^{\mathrm{es}}_c := \bigoplus_{\gamma} \sum_{l\in L^{\mathrm{es}}_{c,\gamma}} \prod_{j=1}^{n} \frac{x_j^{l_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi \mathrm{i}})}$$

and the *non-essential part* to be $\Gamma_c - \Gamma^{\mathrm{es}}_c$.

Henceforth we will add superscripts $\pm$ to the notations defined above to distinguish between Gamma series associated to different triangulations $\Sigma_\pm$.

The main goal of this section is to compute the analytic continuation of the Gamma series solution Γ^+ to bbGKZ$(C,0)$ defined on U_{Σ_+} along the following path (see Figure 3.1) to U_{Σ_-}:

- The start and end points $x_\pm \in U_{\Sigma_\pm}$ should be chosen so that both of them lie in the open set $U_{\Sigma_+\cap\Sigma_-}$[1] and satisfy $\arg(x_+)_j = \arg(x_-)_j$ and $-\log|(x_+)_j| + \log|(x_-)_j| = Ah_j$ for any j and some constant $A > 0$.[2]
- The path $x(u)$, $0 \le u \le 1$ from x_+ to x_- is chosen so that for any $u \in [0,1]$

$$\arg(x(u)_j) = \arg(x_+)_j = \arg(x_-)_j,$$
$$\log|x(u)_j| = (1-u)\log|(x_+)_j| + u\log|(x_-)_j|.$$

  Moreover, we require the argument of the following auxiliary variable

$$y := e^{\mathrm{i}\pi\sum_{j\in I_-} h_j}\prod_{j=1}^{n} x_j^{h_j}$$

  is restricted in the interval $(-2\pi, 0)$ along the path. The existence of this path is guaranteed by the choice of $x_\pm$.

Remark 43. The restriction on the argument of the variable y is imposed to avoid introducing monodromy during the process of analytic continuation.

The main idea comes from [10] where the technique of Mellin-Barnes integrals is used to compute the analytic continuation for the usual GKZ systems. The main difference is that while they worked with K-theory-valued solutions, we work with the

[1]Here $\Sigma_+\cap\Sigma_-$ denotes the set of common maximal cones of $\Sigma_\pm$, see Remark 42.

[2]This condition is equivalent to say that the line connecting $\log|(x_+)_j|$ and $\log|(x_-)_j|$ is perpendicular to the wall that separates Σ_+ and Σ_- in the secondary fan.

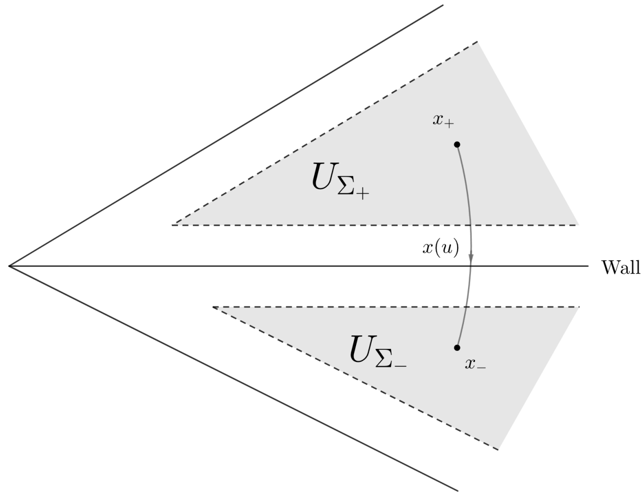

Figure 3.1: Path of the analytic continuation

orbifold cohomology-valued solutions which makes the computation simpler, inspired by the approach of [13].

Remark 44. Let us make a remark here that throughout the remaining of this section, we think of the symbols D_i's as *generic complex numbers*. The reason will be clear once we arrive at the proof of Theorem 54.

In the following we study the analytic continuation of the essential part and the non-essential part separately. The latter case is easier.

Proposition 45. The analytic continuation of $\Gamma_c^+ - \Gamma_c^{+,\mathrm{es}}$ along the path λ is equal to $\Gamma_c^- - \Gamma_c^{-,\mathrm{es}}$.

Proof. By definition each single term $\prod_{j=1}^n \frac{x_j^{l_j+\frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi\mathrm{i}})}$ in the non-essential part $\Gamma_c^\pm - \Gamma_c^{\pm,\mathrm{es}}$ corresponds to some $l \in \bigcup_{\sigma\in\Sigma_+\cap\Sigma_-} L_{c,\gamma,\sigma}$. According to the choice of $x_\pm$ both of them lie in $U_{\Sigma_+\cap\Sigma_-}$, the convexity then assures that the whole path $x(u)$ is contained in this open set. Therefore by Proposition 41 the non-essential parts $\Gamma_c^\pm - \Gamma_c^{\pm,\mathrm{es}}$ are analytic on a open set which contains the analytic continuation path. This finishes the proof. $\square$

The rest of this section will be devoted to the continuation of the essential part $\Gamma_c^{+,\mathrm{es}}$. From now on we fix a twisted sector $\gamma \in \mathrm{Box}(\Sigma_+)$ such that $L_{c,\gamma}$ is non-empty[3] and look at the corresponding component of the Gamma series.

Recall from Section 2.2 that h is the primitive integral linear relation associated to the wall-crossing from Σ_+ to Σ_-. It's clear from the definition that h acts on the lattice $L_{c,\gamma}^+$ by translation. It's also clear that if $l \in L_{c,\gamma}^{\mathrm{es},+}$ then for any $m \geq 0$, the translation $l + mh$ also lies in $L_{c,\gamma}^{\mathrm{es},+}$ (see [10, Proposition 4.7] for a similar result for usual GKZ systems). Hence the following subset of $L_{c,\gamma}^{\mathrm{es},+}$

$$\tilde{L}_{c,\gamma}^{\mathrm{es},+} := \{l \in L_{c,\gamma}^{\mathrm{es},+} : l - h \notin L_{c,\gamma}^{\mathrm{es},+}\}$$

is well-defined, and $L_{c,\gamma}^{\mathrm{es},+} = \tilde{L}_{c,\gamma}^{\mathrm{es},+} + \mathbb{Z}_{\geq 0} h$. We can define $\tilde{L}_{c,\gamma'}^{\mathrm{es},-}$ in the same way with an appropriate change of signs.

With this notation, the essential part $\Gamma_{c,\gamma}^{+,\mathrm{es}}$ is equal to

$$\sum_{l \in L_{c,\gamma}^{\mathrm{es},+}} \prod_{j=1}^{n} \frac{x_j^{l_j + \frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1 + l_j + \frac{D_j}{2\pi \mathrm{i}})} = \sum_{l' \in \tilde{L}_{c,\gamma}^{\mathrm{es},+}} \sum_{m=0}^{\infty} \prod_{j=1}^{n} \frac{x_j^{l'_j + mh_j + \frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1 + l'_j + mh_j + \frac{D_j}{2\pi \mathrm{i}})}.$$

By applying the Euler identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we can rewrite the product as

$$\prod_{j=1}^{n} x_j^{l'_j + \frac{D_j}{2\pi \mathrm{i}}} \cdot \frac{\prod_{j \in I_-} \frac{\sin(\pi(-l'_j - \frac{D_j}{2\pi \mathrm{i}}))}{\pi} \Gamma(-l'_j - mh_j - \frac{D_j}{2\pi \mathrm{i}})}{\prod_{j \notin I_-} \Gamma(1 + l'_j + mh_j + \frac{D_j}{2\pi \mathrm{i}})} \cdot \left((-1)^{\sum_{j \in I_-} h_j} \prod_{i=1}^{n} x_j^{h_j} \right)^m.$$

Now we consider

$$I(s) = -\prod_{j=1}^{n} x_j^{l'_j + \frac{D_j}{2\pi \mathrm{i}}} \frac{\prod_{j \in I_-} \frac{\sin(\pi(-l'_j - \frac{D_j}{2\pi \mathrm{i}}))}{\pi} \Gamma(-l'_j - sh_j - \frac{D_j}{2\pi \mathrm{i}})}{\prod_{j \notin I_-} \Gamma(1 + l'_j + sh_j + \frac{D_j}{2\pi \mathrm{i}})} \Gamma(-s)\Gamma(1+s) \left(e^{\mathrm{i}\pi} y\right)^s$$

[3] If the set $L_{c,\gamma}$ is empty, then the corresponding component $\Gamma_{c,\gamma}$ is equal to zero, thus there is no need for analytic continuation.

where $y = e^{\mathrm{i}\pi \sum_{j\in I_-} h_j} \prod_{j=1}^n x_j^{h_j}$. There are two types of poles of $I(s)$:

- Integers $s = m \in \mathbb{Z}$, which come from the factor $\Gamma(-s)\Gamma(1+s)$.
- $s = p_{k,w} := -\frac{1}{h_k}\left(l'_k + \frac{D_k}{2\pi\mathrm{i}}\right) + \frac{w}{h_k}$, $k \in I_-$, $w \in \mathbb{Z}_{\geq 0}$, which come from the factor $\prod_{j\in I_-} \Gamma(-l'_j - sh_j - \frac{D_j}{2\pi\mathrm{i}})$.

The reason why we use this specific form of $I(s)$ is that the residue of $I(s)$ at integers $s = m \in \mathbb{Z}$ is exactly

$$\operatorname{Res}_{s=m\in\mathbb{Z}} I(s) = \prod_{j=1}^n \frac{x_j^{l'_j + mh_j + \frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1 + l'_j + mh_j + \frac{D_j}{2\pi\mathrm{i}})}$$

that is what we have in the original Gamma series. Also note that the residues at non-positive integers are in fact zero, which follows directly from the definitions of l' and $\tilde{L}^{\mathrm{es},+}_{c,\gamma}$.

The next step is to compute the residues of $I(s)$ at $p_{k,w}$. An application of the Euler identity together with an elementary computation show that

$$\begin{aligned} I(s) &= -\frac{\pi e^{\mathrm{i}\pi s}}{\sin(-\pi s)} \prod_{j\in I_-} \frac{\sin(\pi(-l'_j - \frac{D_j}{2\pi\mathrm{i}}))}{\sin(\pi(-l'_j - sh_j - \frac{D_j}{2\pi\mathrm{i}}))} e^{\mathrm{i}\pi(\sum_{j\in I_-} h_j)s} \prod_{j=1}^n \frac{x_j^{l'_j + sh_j + \frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1 + l'_j + sh_j + \frac{D_j}{2\pi\mathrm{i}})} \\ &= \frac{2\pi\mathrm{i}}{1 - e^{-2\mathrm{i}\pi s}} \prod_{j\in I_-} \frac{1 - e^{-2\mathrm{i}\pi(l'_j + \frac{D_j}{2\pi\mathrm{i}})}}{1 - e^{-2\mathrm{i}\pi(l'_j + sh_j + \frac{D_j}{2\pi\mathrm{i}})}} \prod_{j=1}^n \frac{x_j^{l'_j + sh_j + \frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1 + l'_j + sh_j + \frac{D_j}{2\pi\mathrm{i}})}. \end{aligned}$$

It suffices to look at the factor $1/(1 - e^{-2\mathrm{i}\pi(l'_k + sh_k + \frac{D_k}{2\pi\mathrm{i}})})$ whose residue at $p_{k,w}$ is equal to $\frac{1}{2\pi\mathrm{i}h_k}$. Putting all these together we get

$$\begin{aligned} \operatorname{Res}_{s=p_{k,w}} I(s) =& \frac{1 - e^{-2\mathrm{i}\pi(l'_k + \frac{D_k}{2\pi\mathrm{i}})}}{h_k(1 - e^{-2\mathrm{i}\pi p_{k,w}})} \prod_{\substack{j\in I_- \\ j\neq k}} \frac{1 - e^{-2\mathrm{i}\pi(l'_j + \frac{D_j}{2\pi\mathrm{i}})}}{1 - e^{-2\mathrm{i}\pi(l'_j + p_{k,w}h_j + \frac{D_j}{2\pi\mathrm{i}})}} \\ &\cdot \prod_{j=1}^n \frac{x_j^{l'_j + p_{k,w}h_j + \frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1 + l'_j + p_{k,w}h_j + \frac{D_j}{2\pi\mathrm{i}})}. \end{aligned}$$

Now we introduce a new notation. We write $w = w_0 \cdot (-h_k) + r$, where $w_0, r \in \mathbb{Z}$ and $0 \le r < -h_k$, and define

$$l'' := l' + \frac{l'_k - r}{-h_k} h.$$

Then we have

$$\begin{aligned} 1 + l'_j + p_{k,w} h_j + \frac{D_j}{2\pi \mathrm{i}} &= 1 + (l'_j - \frac{l'_k}{h_k} h_j + w \frac{h_j}{h_k}) + (\frac{D_j - \frac{h_j}{h_k} D_k}{2\pi \mathrm{i}}) \\ &= 1 + \left(l''_j - w_0 h_j\right) + (\frac{D_j - \frac{h_j}{h_k} D_k}{2\pi \mathrm{i}}). \end{aligned}$$

We denote the associated twisted sector by $\gamma^{(k,r)} \in \mathrm{Box}(\Sigma_-^{\mathrm{es}})$. The key observation is $l'' \in \tilde{L}^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$.

Lemma 46. Given a $l' \in \tilde{L}^{\mathrm{es},+}_{c,\gamma}$, $k \in I_-$ and $0 \le r < -h_k$, there is a uniquely determined essential twisted sector $\gamma^{(k,r)} \in \mathrm{Box}(\Sigma_-)^{\mathrm{es}}$ such that $l'' \in \tilde{L}^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$.

Proof. The twisted sector $\gamma^{(k,r)}$ is defined as the associated twisted sector of l'' in the sense of Lemma 18, i.e., $\gamma^{(k,r)} := \sum_{j=1}^n \{l''_j\} v_j$.

First we show that $\gamma^{(k,r)}$ is an essential twisted sector in Σ_-. This is equivalent to show that $\{j : l''_j \notin \mathbb{Z}\}$ is a subcone of an essential cone in Σ_-. Since $l' \in \tilde{L}^{\mathrm{es},+}_{c,\gamma}$, we can write

$$I(l') = \{j : l'_j \notin \mathbb{Z}_{\ge 0}\} \subseteq \mathcal{F} \sqcup I \backslash \{i\} \in \Sigma_+^{\mathrm{es}}$$

for some separated set $\mathcal{F}$ and $i \in I_+$. Take j such that $l''_j \notin \mathbb{Z}$. If $j \notin I$, then $h_j = 0$ and $l''_j = l'_j$, therefore $j \in I(l')$, hence $j \in \mathcal{F}$. On the other hand, if $j \in I$, then $j \neq k$ because by the definition of l'' we have $l''_k = r \in \mathbb{Z}_{\ge 0}$ therefore $k \notin I(l'')$. Now we conclude that $I(l'') \subseteq \mathcal{F} \sqcup I \backslash \{k\}$, which is an essential cone in Σ_- because $k \in I_-$. This also shows that l'' lies in $L^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$.

Next we show that $l'' \in \tilde{L}^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$. It suffices to show that $l'' + h \notin L^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$. This follows from the construction of l''. Note that l'' is chosen such that $l''_k = r \in \mathbb{Z}_{\geq 0}$ while $l''_k + h_k = r + h_k \in \mathbb{Z}_{<0}$, this implies that $I(l'') \subseteq \mathcal{F} \sqcup I \backslash \{k\}$ while $I(l''+h) \nsubseteq \mathcal{F} \sqcup I \backslash \{k\}$. Therefore $l'' + h \notin L^{\mathrm{es},-}_{c,\gamma^{(k,r)}}$ and the proof is completed. $\square$

The first two terms in the residue can be written as

$$\frac{1 - e^{-2\mathrm{i}\pi(l'_k + \frac{D_k}{2\pi\mathrm{i}})}}{h_k(1 - e^{-2\mathrm{i}\pi(\frac{l'_k - l''_k}{-h_k} - \frac{D_k}{2\pi\mathrm{i}h_k})})} \prod_{\substack{j \in I_- \\ j \neq k}} \frac{1 - e^{-2\mathrm{i}\pi(l'_j + \frac{D_j}{2\pi\mathrm{i}})}}{1 - e^{-2\mathrm{i}\pi(l''_j + \frac{D_j - \frac{h_j}{h_k} D_k}{2\pi\mathrm{i}})}}.$$

We claim that this factor only depends on the twisted sectors γ and $\gamma^{(k,r)}$. To see this, recall our choice of the lifting $\mathrm{Box}(\Sigma_\pm) \to \mathbb{K}_c$ made in Remark 20. It's then clear that $(l' - l'') - (\gamma - \gamma^{(k,r)})$ is also a rational multiple of h. However the left side has integral coordinates, which forces the right side to be a integer multiple of h due to the primitivity. This implies that the k-th coordinate $(l'_k - l''_k) - (\gamma_k - \gamma_k^{(k,r)})$ is an integer multiple of h_k, which means

$$1 - e^{-2\mathrm{i}\pi(\frac{l'_k - l''_k}{-h_k} - \frac{D_k}{2\pi\mathrm{i}h_k})} = 1 - e^{-2\mathrm{i}\pi(\frac{\gamma_k - \gamma_k^{(k,r)}}{-h_k} - \frac{D_k}{2\pi\mathrm{i}h_k})}.$$

Together with the facts that $l'_j \equiv \gamma_j$ and $l''_j \equiv \gamma_j^{(k,r)}$ modulo $\mathbb{Z}$, the original factor is then equal to

$$C_{\gamma^{(k,r)}} := \frac{1 - e^{-2\mathrm{i}\pi(\gamma_k + \frac{D_k}{2\pi\mathrm{i}})}}{h_k(1 - e^{-2\mathrm{i}\pi(\frac{\gamma_k - \gamma_k^{(k,r)}}{-h_k} - \frac{D_k}{2\pi\mathrm{i}h_k})})} \prod_{\substack{j \in I_- \\ j \neq k}} \frac{1 - e^{-2\mathrm{i}\pi(\gamma_j + \frac{D_j}{2\pi\mathrm{i}})}}{1 - e^{-2\mathrm{i}\pi(\gamma_j^{(k,r)} + \frac{D_j - \frac{h_j}{h_k} D_k}{2\pi\mathrm{i}})}}$$

hence the residue is

$$\mathrm{Res}_{s=p_{k,w}} I(s) = C_{\gamma^{(k,r)}} \cdot \prod_{j=1}^{n} \frac{x_j^{(l''_j - w_0 h_j) + (\frac{D_j - \frac{h_j}{h_k} D_k}{2\pi\mathrm{i}})}}{\Gamma(1 + (l''_j - w_0 h_j) + \frac{D_j - \frac{h_j}{h_k} D_k}{2\pi\mathrm{i}})}.$$

Finally, we use the techniques of Mellin-Barnes integrals to finish the computation. First of all we fix a $l' \in \tilde{L}^{\text{es},+}_{c,\gamma}$ and do the analytic continuation to the corresponding subseries

$$\sum_{m=0}^{\infty}\prod_{j=1}^{n}\frac{x_j^{l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}})}.$$

We consider the contour integral

$$\frac{1}{2\pi \mathrm{i}}\int_C I(s)ds = \frac{1}{2\pi \mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} I(s)ds$$

Here the contour C is parallel to the imaginary axis, and the real part a of C is a negative number satisfying $\epsilon < |a| < 1$ for some $\epsilon > 0$ which avoids any pole of the integrand.

Now by [10, Lemma A.6] (see also the proof of [10, Theorem 4.10]), the sum of residues of $I(s)$ at the poles on the right side of the contour C

$$\sum_{m=0}^{\infty}\prod_{j=1}^{n}\frac{x_j^{l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}})} + \sum_{p_{k,w}\text{ on the right side of }C} \mathrm{Res}_{p_{k,w}} I(s) \tag{3.1.1}$$

is analytically continued to

$$-\sum_{p_{k,w}\text{ on the left side of }C} \mathrm{Res}_{p_{k,w}} I(s) \tag{3.1.2}$$

that is the negative of the sum of residues of $I(s)$ at poles on the left side of C. Note that for a fixed $l' \in \tilde{L}^{\text{es},+}_{c,\gamma}$, the real part of the poles $p_{k,w}$ is bounded above, therefore the second sum in (3.1.1) is finite. Therefore we can add the negative of it to both of

(3.1.1) and (3.1.2), and deduce that

$$\sum_{m=0}^{\infty}\prod_{j=1}^{n}\frac{x_j^{l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l'_j+mh_j+\frac{D_j}{2\pi \mathrm{i}})}$$

is analytically continued to

$$-\sum_{k\in I_-}\sum_{0\leq r<-h_k} C_{\gamma(k,r)}\sum_{w_0=0}^{\infty}\prod_{j=1}^{n}\frac{x_j^{(l''_j-w_0h_j)+(\frac{D_j-\frac{h_j}{h_k}D_k}{2\pi \mathrm{i}})}}{\Gamma(1+(l''_j-w_0h_j)+\frac{D_j-\frac{h_j}{h_k}D_k}{2\pi \mathrm{i}})}$$

To proceed with the analytic continuation, we need some analytic results and estimates. The corresponding results in the setting of the usual GKZ systems could be found in [10]. Indeed, the results in this appendix could be proved word for word following the argument therein.

We denote the intersection of the cone C_{Σ_+} and C_{Σ_-} in the secondary fan (i.e., the wall defined by the linear relation h) by $\widetilde{C}$.

Lemma 47. For any $k, A > 0$ there exists $\widetilde{c}$ in the interior of $\widetilde{C}$ such that for any $l' \in \widetilde{L}^{\mathrm{es},+}_{c,\gamma}$ we have

$$\sum_{j=1}^{n} l'_j u_j \geq k\|l'\|$$

for any $u \in \widetilde{C}+\widetilde{c}+a$ and any $a \in \mathbb{R}^n$ with $\|a\| \leq A$.

Proof. The proof is the same with the proof of [10, Lemma 4.11], the only difference is that the $\widetilde{L}^{\mathrm{es},+}_{c,\gamma}$ is a shift of the $\mathcal{S}'$ therein. See also the proof of [5, Proposition 3.8]. □

Lemma 48. There exists $A > 0$ and $\widetilde{c} \in \widetilde{C}$ such that the set

$$V_A := \bigcup_{a:\|a\|<A}(\widetilde{C}+\widetilde{c}+a)$$

intersects with $U_{\Sigma_\pm}$, and such that the integral[4]

$$\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \sum_{l' \in \widetilde{L}_{c,\gamma}^{\mathrm{es},+}} I_{l'}(s) ds$$

is absolutely convergent on the region U defined by

$$U = \{(x_j) \in \mathbb{C}^n : (-\log |x_j|) \in V_A, \quad -2\pi < \arg y < 0,\ \arg(\mathbf{x}) \in (-\pi, \pi)^n\}$$

Proof. The proof is parallel to the proof of [10, Lemma 4.12]. The contour is defined by $s = a + \mathrm{i}t$ for $t \in \mathbb{R}$. By applying [10, Lemma A.5] we see that the integrand $I_{l'}(s)$ is controlled by

$$|y|^a e^{-(\pi + \arg y)t}(|t| + 1)^{R+n/2} e^{-\pi|t|} \sum_{l' \in \widetilde{L}_{c,\gamma}^{\mathrm{es},+}} (4ek)^{\|l'\|} e^{\sum l'_j \log |x_j|}$$

for some $R > 0$ independent of l'. Now apply Lemma 47, we can choose $\widetilde{c}$ in the interior $\widetilde{C}$ such that on the set V_A we have

$$(4ek)^{\|l'\|} e^{\sum l'_j \log |x_j|} \leq e^{-\epsilon \|l'\|}$$

for some $\epsilon > 0$ and any $l' \in \widetilde{L}_{c,\gamma}^{\mathrm{es},+}$ and $x \in U$. Since $\|l'\|$ is of polynomial growth, the sum over all l' is still controlled by an exponential function with negative exponent. Hence the integral is absolutely convergent and therefore defines an analytic function over U. □

Now Lemma 48 together with the fact that the sum of the second term in (4.2.1) over all $l' \in \tilde{L}_{c,\gamma}^{\mathrm{es},+}$ is absolutely convergent (because it is a subseries of a finite sum of the original gamma series) allows us to deduce the following desired analytic contin-

[4]Here we use the subscript l' to emphasize the dependence of the integrand on l'.

uation of $\Gamma^{+}_{c,\gamma}$

$$-\sum_{k\in I_-}\sum_{0\le r<-h_k} C_{\gamma(k,r)} \sum_{l''\in\tilde{L}^{\mathrm{es},-}_{c,\gamma(k,r)}} \sum_{w_0=0}^{\infty}\prod_{j=1}^{n} \frac{x_j^{(l''_j-w_0h_j)+(\frac{D_j-\frac{h_j}{h_k}D_k}{2\pi\mathrm{i}})}}{\Gamma(1+(l''_j-w_0h_j)+\frac{D_j-\frac{h_j}{h_k}D_k}{2\pi\mathrm{i}})}$$

that is

$$-\sum_{k\in I_-}\sum_{0\le r<-h_k} C_{\gamma(k,r)}\Gamma^{-,\mathrm{es}}_{c,\gamma(k,r)}|_{D_j\to D_j-\frac{h_j}{h_k}D_k}$$

where the subscript of $\Gamma^{-,\mathrm{es}}_{c,\gamma(k,r)}|_{D_j\to D_j-\frac{h_j}{h_k}D_k}$ denotes substitution of D_j by $D_j-\frac{h_j}{h_k}D_k$, and we have used the fact $l''\in\tilde{L}^{\mathrm{es},-}_{c,\gamma(k,r)}$. Therefore we have proved the following result.

Proposition 49. The analytic continuation of $\Gamma^{+,\mathrm{es}}_{c,\gamma}$ is given by

$$-\sum_{k\in I_-}\sum_{0\le r<-h_k} C_{\gamma(k,r)}\Gamma^{-,\mathrm{es}}_{c,\gamma(k,r)}|_{D_j\to D_j-\frac{h_j}{h_k}D_k}$$

3.2 Fourier-Mukai transforms

In this section we compute the Fourier-Mukai transform associated to the toric wall-crossing $\mathbb{P}_{\Sigma_-}\dashrightarrow\mathbb{P}_{\Sigma_+}$ and match it with the analytic continuation computed in Section 3.1.

The Fourier-Mukai transform associated to the flop $\mathbb{P}_{\Sigma_-}\dashrightarrow\mathbb{P}_{\Sigma_+}$ was studied by Borisov and Horja in [11] and [10]. More precisely, they computed the images of the K-theory classes R_i of $\mathbb{P}_{\Sigma_-}$ under the pullback and pushforward functors and obtained the following formulae for the K-theoretic Fourier-Mukai transforms. See [10, Proposition 5.1, 5.2].

Before we state their result, we introduce some notations first. Let $\gamma_+=\sum_j(\gamma_+)_jv_j$ be an essential twisted sector of Σ_+. We denote by $\mathcal{I}(\gamma_+)$ the set of complex numbers t such that $e^{2\pi\mathrm{i}(\gamma_+)_j}\cdot t^{h_j}=1$ for some $j\in I_-$. It is proved in [10, Section 4] that this set is finite, consists of roots of unity, and is in 1-1 correspondence with the adjacent twisted sectors of γ_+.

Proposition 50. (1) For any analytic function φ and J which is not a subcone of any essential cone, the class

$$\prod_{j\in J}(1-R_j)\varphi(R)$$

remains unchanged under the Fourier-Mukai transform.

(2)For any analytic function φ the image of the K-theory class $\varphi(R)=\varphi(R_1,\cdots,R_n)$ under the Fourier-Mukai transform FM is given by

$$\mathrm{FM}(\varphi(R)) = (\mathrm{FM}(\varphi)(\mathcal{R}))(1)$$

where the function $\mathrm{FM}(\varphi)$ is defined as

$$\mathrm{FM}(\varphi)(r) = \varphi(r) - \sum_{t\in\mathcal{I}}\int_{C_t} T(r,\hat{t})\varphi(r\hat{t}^h)d\hat{t}$$

where $T(r,\hat{t}) = \frac{1}{2\pi\mathrm{i}(\hat{t}-1)}\prod_{j\in I_-}\frac{1-r_j^{-1}}{1-r_j^{-1}\hat{t}^{-h_j}}$, $\mathcal{I}$ is a set of roots of unity of a large enough order such that it contains $\mathcal{I}(\gamma_+)$ defined above for all essential twisted sectors γ_+ of Σ_+. The contours C_t for $t\in\mathcal{I}$ are circles defined in a way such that they include all poles of the integrand in the interior, and $\mathcal{R}_i$ is the endomorphism on $K_0(\mathbb{P}_{\Sigma_+})$ defined by multiplication by R_i.

In this section we use the formulae of Borisov-Horja to compute the Fourier-Mukai transform of the non-essential part $\Gamma_c^- - \Gamma_c^{-,\mathrm{es}}$ and the essential part $\Gamma_c^{-,\mathrm{es}}$ separately. A comparison of the computation in this section with the one in the last section hence yields the FM=AC result for bbGKZ$(C,0)$.

The non-essential part is easier to deal with.

Proposition 51. The Fourier-Mukai transform $\mathrm{FM}(\Gamma_c^- - \Gamma_c^{-,\mathrm{es}})$ is given by

$$\mathrm{FM}(\Gamma_c^- - \Gamma_c^{-,\mathrm{es}}) = \Gamma_c^- - \Gamma_c^{-,\mathrm{es}}$$

Proof. By definition for each single term $\prod_{j=1}^n \frac{x_j^{l_j+\frac{D_j}{2\pi \mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi \mathrm{i}})}$ in the non-essential part $\Gamma_c^- - \Gamma_c^{-,\mathrm{es}}$, the set $I(l)$ (therefore $\{i : l_i \in \mathbb{Z}_{<0}\}$) is not a subcone of any essential cone. Hence it contributes a factor of the form $\prod_{j\in J} D_j$ where J is not a subcone of any essential cone. Note that D_j can be written as the product of $1-e^{D_j}$ with an invertible element, the original product can be therefore written as

$$\prod_{j\in J}(1-e^{D_j})\widetilde{\varphi}(D)$$

where $\widetilde{\varphi}$ is an analytic function. Taking direct sum over all twisted sectors γ, we see that under the Chern character $K_0(\mathbb{P}_{\Sigma_-}) \xrightarrow{\sim} \oplus_\gamma H_\gamma$ the non-essential part $\Gamma_c^- - \Gamma_c^{-,\mathrm{es}}$ is exactly of the form $\prod_{j\in J}(1-R_j)\varphi(R)$ for some analytic function φ. Now the statement follows from the first part of Proposition 50. □

In order to compare the Fourier-Mukai transform of the essential part $\Gamma_c^{-,\mathrm{es}}$ with the analytic continuation computed in the last section, we first rewrite the formula in the second part of Proposition 50 by computing the residue of the integrand explicitly. Recall that $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ is a semi-local ring whose maximal ideals are in 1-1 correspondence with twisted sectors $\gamma \in \mathrm{Box}(\Sigma)$.

Proposition 52. For any analytic function φ we have

$$\mathrm{FM}(\varphi)(z)_\gamma = \begin{cases} \varphi(z), \text{ if } \gamma \notin \mathrm{Box}(\Sigma_+)^{\mathrm{es}} \\ -\sum_{k\in I_-}\sum_{0\le r<-h_k} C_{\gamma^{(k,r)}}\varphi(z(p^{(k,r)})^h), \text{ if } \gamma \in \mathrm{Box}(\Sigma_+)^{\mathrm{es}} \end{cases}$$

where $p^{(k,r)} := e^{-2\mathrm{i}\pi(\frac{\gamma_k-\gamma_k^{(k,r)}}{-h_k}-\frac{D_k}{2\pi \mathrm{i}h_k})}$ for $k \in I_-$, $0 \le r < -h_k$.

Proof. Fix a twisted sector $\gamma \in \mathrm{Box}(\Sigma_+)$, we localize at the point $r_j = e^{D_j + 2\pi \mathrm{i} \gamma_j}$ corresponding to γ. If $\gamma \notin \mathrm{Box}(\Sigma_+)^{\mathrm{es}}$, then by definition of $\mathcal{I}$ the integration kernel $T(r, \hat{t})$ has no pole inside the contours C_t, therefore the second term in the formula is equal to zero.

Now suppose $\gamma \in \mathrm{Box}(\Sigma_+)^{\mathrm{es}}$. The poles of $T(r, \hat{t})$ are 1 together with $\hat{t}$'s such that there exists $k \in I_-$ with $r_k \hat{t}^{h_k} = 1$ where $r_k = e^{D_k + 2\pi \mathrm{i} \gamma_k}$. The set of poles is then

$$\left\{ e^{-\frac{1}{h_k}(2\pi \mathrm{i} \gamma_k + D_k)} (e^{\frac{2\pi \mathrm{i}}{h_k}})^r \ : \ k \in I_-, 0 \le r < -h_k \right\}.$$

An elementary calculation shows that this set is in fact the same as

$$\left\{ p^{(k,r)} := e^{-2\mathrm{i}\pi(\frac{\gamma_k - \gamma_k^{(k,r)}}{-h_k} - \frac{D_k}{2\pi \mathrm{i} h_k})} \ : \ k \in I_-, 0 \le r < -h_k \right\}$$

where $\gamma^{(k,r)}$ is defined as in the previous section[5]. To demonstrate this, we observe that according to Remark 21, any two liftings differ by an integer multiple of h. Thus, $p^{(k,r)}$ does not depend on the choice of lifting, and the set $\{\gamma_k^{(k,r)}\}$ is equal to $\{0, 1, \cdots, -h_k - 1\}$ modulo $-h_k$. To compute the residue of $T(r, \hat{t})$ at these poles, it suffices to consider the factor $\frac{1 - r_k^{-1}}{1 - r_k^{-1} \hat{t}^{-h_k}}$. The residue is then equal to

$$\begin{aligned}
2\pi \mathrm{i} \, \mathrm{Res}_{\hat{t} = p^{(r,k)}} T(r, \hat{t}) &= \frac{1}{p^{(r,k)} - 1} \prod_{\substack{j \in I_- \\ j \neq k}} \frac{1 - e^{-D_j - 2\pi \mathrm{i} \gamma_j}}{1 - e^{-D_j - 2\pi \mathrm{i} \gamma_j} (p^{(r,k)})^{-h_j}} \cdot \frac{p^{(r,k)}(1 - r_k^{-1})}{h_k} \\
&= \frac{e^{2\pi \mathrm{i}(-\gamma_k - \frac{D_k}{2\pi \mathrm{i}})} - 1}{h_k (e^{\frac{2\mathrm{i}\pi}{-h_k}(\gamma_k^{(r,k)} - \gamma_k - \frac{D_k}{2\pi \mathrm{i}})} - 1)} \prod_{\substack{j \in I_- \\ j \neq k}} \frac{1 - e^{-2\mathrm{i}\pi(\gamma_j + \frac{D_j}{2\pi \mathrm{i}})}}{1 - e^{-2\mathrm{i}\pi(\gamma_j^{(r,k)} + \frac{D_j - \frac{h_j}{h_k} D_k}{2\pi \mathrm{i}})}} \\
&= C_{\gamma^{(k,r)}}.
\end{aligned}$$

Putting all these together we obtain the desired result. ∎

[5]We note that the $\gamma_k^{(k,r)}$ denotes the k-th coordinate of the *lifting we chose for* $\gamma^{(k,r)}$ in the sense of Lemma 20, not necessarily equal to $\{l''_k\}$ which is zero.

Corollary 53. The Fourier-Mukai transform $\mathrm{FM}(\Gamma_c^{-,\mathrm{es}})$ is given by

$$\mathrm{FM}(\Gamma_c^{-,\mathrm{es}})_\gamma = -\sum_{k\in I_-}\sum_{0\le r<-h_k} C_{\gamma(k,r)}\Gamma^-_{c,\gamma(k,r)}|_{D_j\to D_j-\frac{h_j}{h_k}D_k}$$

for each twisted sector $\gamma \in \mathrm{Box}(\Sigma_+)^{\mathrm{es}}$.

Proof. Apply Proposition 52 to $\Gamma_c^{-,\mathrm{es}}$. □

Theorem 54. *The following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathbb{P}_{\Sigma_+})^\vee & \xrightarrow{-\circ\Gamma_+} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_+)) \\ \downarrow{\scriptstyle \mathrm{FM}^\vee} & & \downarrow{\scriptstyle \mathrm{MB}} \\ K_0(\mathbb{P}_{\Sigma_-})^\vee & \xrightarrow{-\circ\Gamma_-} & \mathrm{Sol}(\mathrm{bbGKZ}(C,U_-)) \end{array}$$

Proof. This is equivalent to prove that for any linear function $\varphi : K_0(\mathbb{P}_{\Sigma_+}) \to \mathbb{C}$ there holds

$$\varphi\circ\mathrm{FM}\circ\Gamma_- = \mathrm{MB}(\varphi\circ\Gamma_+)$$

which follows directly from Proposition 45, Proposition 49, Proposition 51, and Corollary 53. It is important to note that this also explains why, in Section 3.1, we made the assumption that all D_i's are generic complex numbers. □

3.3 Compactly-supported derived categories and dual systems

In this section we make use of the duality result in [5] to prove the analogous result for the dual system $\mathrm{bbGKZ}(C^\circ, 0)$.

Recall from Section 2.4.2 that there is a similarly defined Gamma series solution Γ° with values in the compactly supported orbifold cohomology $H^*_{\mathrm{orb},c} = \bigoplus_\gamma H^c_\gamma$ to the dual system bbGKZ$(C^\circ, 0)$. We define

$$\Gamma_c^\circ(x_1, \ldots, x_n) = \bigoplus_\gamma \sum_{l \in L_{c,\gamma}} \prod_{i=1}^n \frac{x_i^{l_i + \frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1 + l_i + \frac{D_i}{2\pi \mathrm{i}})} \left(\prod_{i \in \sigma} D_i^{-1} \right) F_\sigma$$

where σ is the set of i with $l_i \in \mathbb{Z}_{<0}$ and F_σ's are generators of $H^*_{\mathrm{orb},c}$ as a module over H^*_{orb}.

The main theorem of this section is the following analogous result for the dual system bbGKZ$(C^\circ, 0)$.

Theorem 55. *The following diagram commutes:*

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\Sigma_+})^\vee & \xrightarrow{-\circ\Gamma_+^\circ} & \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ), U_+) \\ \downarrow{\scriptstyle (\mathrm{FM}^c)^\vee} & & \downarrow{\scriptstyle \mathrm{MB}^c} \\ K_0^c(\mathbb{P}_{\Sigma_-})^\vee & \xrightarrow{-\circ\Gamma_-^\circ} & \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ), U_-) \end{array} \tag{3.3.1}$$

First we prove that there is a well-defined compactly supported K-theoretic Fourier-Mukai transform $\mathrm{FM}^c : K_0^c(\mathbb{P}_{\Sigma_-}) \to K_0^c(\mathbb{P}_{\Sigma_+})$.

Lemma 56. Let $f : \mathbb{P}_{\hat{\Sigma}} \to \mathbb{P}_\Sigma$ be the weighted blow-up along the closed substack $\mathbb{P}_{\Sigma/\sigma_I}$, where σ_I is a cone in Σ (not necessarily an interior cone). Then there are

$$f^{-1}(\pi_\Sigma^{-1}(0)) \subseteq \pi_{\hat{\Sigma}}^{-1}(0), \quad f(\pi_{\hat{\Sigma}}^{-1}(0)) \subseteq \pi_\Sigma^{-1}(0)$$

Proof. The structure morphisms are compatible with the blow-up

$$\begin{array}{ccc} \mathbb{P}_{\hat{\Sigma}} & \xrightarrow{f} & \mathbb{P}_{\boldsymbol{\Sigma}} \\ & {\scriptstyle \pi_{\hat{\Sigma}}} \searrow & \downarrow{\scriptstyle \pi_\Sigma} \\ & & \mathrm{Spec}\, \mathbb{C}[C^\vee \cap N^\vee] \end{array}$$

which induces the following Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}_{\hat{\Sigma}} \backslash \pi_{\hat{\Sigma}}^{-1}(0) & \hookrightarrow & \mathbb{P}_{\hat{\Sigma}} \\ \downarrow & & \downarrow f \\ \mathbb{P}_{\mathbf{\Sigma}} \backslash \pi_{\Sigma}^{-1}(0) & \hookrightarrow & \mathbb{P}_{\mathbf{\Sigma}} \end{array}$$

the lemma follows directly from the commutativity of this diagram. □

Theorem 57. *The Fourier-Mukai transform* $\mathrm{FM} : D^{\mathrm{b}}(\mathbb{P}_{\Sigma_-}) \to D^{\mathrm{b}}(\mathbb{P}_{\Sigma_+})$ *maps* $D^{\mathrm{b}}_c(\mathbb{P}_{\Sigma_-})$ *to* $D^{\mathrm{b}}_c(\mathbb{P}_{\Sigma_+})$.

Proof. The center $\mathbb{P}_{\Sigma_-/I_+}$ of the blow up $\mathbb{P}_{\hat{\Sigma}} \to \mathbb{P}_{\Sigma_-}$ can be viewed as the zero locus of a regular section of the vector bundle $E = \bigoplus_{j \in I_+} \mathscr{O}_{\mathbb{P}_{\Sigma_-}}(D_j)$, therefore the blow up f_- can be decomposed as

$$\begin{array}{ccc} \mathbb{P}_{\hat{\Sigma}} & \overset{i}{\hookrightarrow} & \mathbb{P}_{\mathbb{P}_{\Sigma_-}}(E^\vee) \\ & \searrow^{f_-} & \downarrow p \\ & & \mathbb{P}_{\Sigma_-} \end{array}$$

where $\mathbb{P}_{\mathbb{P}_{\Sigma_-}}(E^\vee)$ is the projective bundle associated to $E^\vee$, and p is the projection. This is well-known for varieties, and can be proved for stacks similarly.

Let $\mathscr{F}$ be an arbitrary coherent sheaf on $\mathbb{P}_{\Sigma_-}$ supported on $\pi_{\Sigma_-}^{-1}(0)$. We first prove that $L(f_-)^*(\mathscr{F})$ is supported on $\pi_{\hat{\Sigma}}^{-1}(0)$. Since i is a closed immersion, it suffices to show that $i_* L(f_-)^* \mathscr{F}$ is supported on the image of $\pi_{\hat{\Sigma}}^{-1}(0)$ under i. Note that we have

$$i_* L(f_-)^* \mathscr{F} = i_* L i^* p^* \mathscr{F} \cong p^* \mathscr{F} \otimes^L i_* \mathscr{O}_{\mathbb{P}_{\hat{\Sigma}}}$$

here we used the facts that p is flat, i_* is exact and $R(f_-)_* \mathscr{O}_{\mathbb{P}_{\hat{\Sigma}}} = \mathscr{O}_{\mathbb{P}_{\Sigma_-}}$. From this we have

$$\mathrm{supp}(i_* L(f_-)^* \mathscr{F}) \subseteq \mathrm{supp}(p^* \mathscr{F}) \cap \mathrm{supp}(i_* \mathscr{O}_{\mathbb{P}_{\hat{\Sigma}}}) \subseteq p^{-1}(\mathrm{supp}(\mathscr{F})) \cap i(\mathbb{P}_{\hat{\Sigma}})$$

$$\subseteq p^{-1}(\pi_{\Sigma_-}^{-1}(0)) \cap i(\mathbb{P}_{\hat{\Sigma}})$$

It suffices to show that $p^{-1}(\pi_{\Sigma_-}^{-1}(0)) \cap i(\mathbb{P}_{\hat{\Sigma}}) \subseteq i(\pi_{\hat{\Sigma}}^{-1}(0))$, which is again equivalent to $(f_-)^{-1}(\pi_{\Sigma_-}^{-1}(0)) \subseteq \pi_{\hat{\Sigma}}^{-1}(0)$. Applying Lemma 56 we get the desired result.

Now consider an arbitrary complex $\mathscr{F}^\bullet$ in $D_c^{\mathrm{b}}(\mathbb{P}_{\Sigma_-})$, we argue by induction on the length of $\mathscr{F}^\bullet$. Denote the lowest degree of non-zero cohomology of $\mathscr{F}^\bullet$ by i_0, then there exists a distinguished triangle

$$\mathscr{H}^{i_0}(\mathscr{F}^\bullet)[-i_0] \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^{i_0}(\mathscr{F}^\bullet)[1-i_0]$$

where the i-th cohomology of $\mathscr{G}^\bullet$ are isomorphic to that of $\mathscr{F}^\bullet$ for all $i > i_0$ and the i_0-th cohomology is zero. Apply the derived pull-back we get a distinguished triangle in $D^{\mathrm{b}}(\mathbb{P}_{\hat{\Sigma}})$

$$L(f_-)^*\mathscr{H}^{i_0}(\mathscr{F}^\bullet)[-i_0] \to L(f_-)^*\mathscr{F}^\bullet \to L(f_-)^*\mathscr{G}^\bullet \to L(f_-)^*\mathscr{H}^{i_0}(\mathscr{F}^\bullet)[1-i_0]$$

By induction assumption both $L(f_-)^*\mathscr{H}^{i_0}(\mathscr{F}^\bullet)$ and $L(f_-)^*\mathscr{G}^\bullet$ are supported on $\pi_{\hat{\Sigma}}^{-1}(0)$, taking stalks of this distinguished triangle we get that $L(f_-)^*\mathscr{F}^\bullet$ is also supported on $\pi_{\hat{\Sigma}}^{-1}(0)$.

Next we look at the pushforward $R(f_+)_* : D^{\mathrm{b}}(\mathbb{P}_{\hat{\Sigma}}) \to D^{\mathrm{b}}(\mathbb{P}_{\Sigma_+})$. Take an arbitrary complex $\mathscr{K}^\bullet$ in $D_c^{\mathrm{b}}(\mathbb{P}_{\hat{\Sigma}})$. Consider the following Cartesian diagram used in the proof of the lemma:

$$\begin{array}{ccc} \mathbb{P}_{\hat{\Sigma}} \backslash \pi_{\hat{\Sigma}}^{-1}(0) & \hookrightarrow & \mathbb{P}_{\hat{\Sigma}} \\ \downarrow & & \downarrow f_+ \\ \mathbb{P}_{\Sigma_+} \backslash \pi_{\Sigma_+}^{-1}(0) & \hookrightarrow & \mathbb{P}_{\boldsymbol{\Sigma}} \end{array}$$

Apply the flat base change formula then we see that the restriction of $R(f_+)_*\mathscr{K}^\bullet$

to $\mathbb{P}_{\Sigma_+}\backslash\pi_{\Sigma_+}^{-1}(0)$ is zero, that is, $R(f_+)_*\mathscr{K}^\bullet$ is supported on $\pi_{\Sigma_+}^{-1}(0)$. So $R(f_+)_*$ maps $D_c^{\mathrm{b}}(\mathbb{P}_{\hat{\Sigma}})$ into $D_c^{\mathrm{b}}(\mathbb{P}_{\Sigma_+})$. □

Therefore the Fourier-Mukai transform induces an isomorphism $\mathrm{FM}^c : K_0^c(\mathbb{P}_{\Sigma_-}) \to K_0^c(\mathbb{P}_{\Sigma_+})$ of compactly-supported K-theories. Now we are ready to prove the main theorem of this section.

Proof of Theorem 55. Since FM is an equivalence of categories, we have

$$\begin{aligned} \chi_-([\mathscr{E}_1^\bullet],[\mathscr{E}_2^\bullet]) &= \sum_i (-1)^i \dim \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}_{\Sigma_-})}(\mathscr{E}_1^\bullet, \mathscr{E}_2^\bullet[i]) \\ &= \sum_i (-1)^i \dim \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}_{\Sigma_+})}(\mathrm{FM}(\mathscr{E}_1^\bullet), \mathrm{FM}^c(\mathscr{E}_2^\bullet)[i]) \\ &= \chi_+(\mathrm{FM}([\mathscr{E}_1^\bullet]), \mathrm{FM}^c([\mathscr{E}_2^\bullet])) \end{aligned} \tag{3.3.2}$$

where $\chi_\pm$ denotes the Euler characteristic pairing on $\mathbb{P}_{\Sigma_\pm}$. Hence FM preserves the Euler characteristic pairing.

To proceed, we need the following duality result for the pair of better-behaved GKZ systems, proved in [5, Theorem 2.4, 4.2]. More precisely, there is a non-degenerate pairing between the solution spaces of $\mathrm{bbGKZ}(C,0)$ and $\mathrm{bbGKZ}(C^\circ,0)$

$$\langle -,-\rangle : \mathrm{Sol}(\mathrm{bbGKZ}(C,0)) \times \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ,0)) \to \mathbb{C}$$

that corresponds to the inverse of the Euler characteristic pairing in the large radius limit under the isomorphisms given by the Gamma series solutions.

Now consider the diagram:

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\Sigma_+})^\vee & \xrightarrow{-\circ\Gamma_+^\circ} & \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ), U_+) \\ \big\downarrow{\scriptstyle (\mathrm{FM}^c)^\vee} & & \big\downarrow{\scriptstyle \mathrm{MB}^c} \\ K_0^c(\mathbb{P}_{\Sigma_-})^\vee & \xrightarrow{-\circ\Gamma_-^\circ} & \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ), U_-) \end{array}$$

To finish the proof, it suffices to show $(\mathrm{FM}^c)^\vee(g)\circ\Gamma_-^\circ = \mathrm{MB}^c(g\circ\Gamma_+^\circ)$ for any $g\in K_0^c(\mathbb{P}_{\Sigma_+})^\vee$. Take an arbitrary $f\in K_0(\mathbb{P}_{\Sigma_+})^\vee$, we have

$$\begin{aligned}
\langle(\mathrm{FM})^\vee(f)\circ\Gamma_-, (\mathrm{FM}^c)^\vee(g)\circ\Gamma_-^\circ\rangle &= \chi_-^\vee((\mathrm{FM})^\vee(f), (\mathrm{FM}^c)^\vee(g))\\
&= \chi_+^\vee(f,g)\\
&= \langle f\circ\Gamma_+, g\circ\Gamma_+^\circ\rangle\\
&= \langle \mathrm{MB}(f\circ\Gamma_+), \mathrm{MB}^c(g\circ\Gamma_+^\circ)\rangle\\
&= \langle(\mathrm{FM})^\vee(f)\circ\Gamma_-, \mathrm{MB}^c(g\circ\Gamma_+^\circ)\rangle.
\end{aligned}$$

Here the first and third equalities follows from the duality of bbGKZ systems, the second equality follows from (3.3.2), the fourth equality follows from the definition of analytic continuation and the last equality is Theorem 54.

Since the pairing of solutions is nondegenerate and $(\mathrm{FM})^\vee(f)\circ\Gamma_-$ span the whole solution space, we obtain the desired result. □

Remark 58. The main result in this chapter shows that certain monodromy (given by analytic continuation along certain "loops" in the stringy Kähler moduli space) of the local systems of solutions to bbGKZ systems matches with the monodromy on the derived categories (given by twisting by line bundles). It would be very interesting to match other types of monodromy, e.g. spherical twists on the derived categories that are not just twisting by line bundles, and analytic continuation along loops around other components of the GKZ discriminant locus. This question seems to be difficult since the standard Mellin-Barnes integral technique will no long work in this general situation.

Chapter 4

Local mirror symmetry and integral structures

In [19], Hosono studied local mirror symmetry (i.e., Hori-Vafa mirrors) in 2- and 3-dimensional cases. We briefly review one of the examples therein.

Example 59 (2-dimensional cases)**.** Consider the A_n-singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$ and its minimal resolution X. Hosono considered the following mirror of X:

$$Y = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* : u^2 + v^2 + f(w) = 0\}$$

where $f(w) = a_0 + a_1 w + \cdots + a_{n+1} w^{n+1}$. The A-brane central charges are defined as

$$\int_\gamma \frac{1}{u^2 + v^2 + f(a; w)} du dv \frac{dw}{w}$$

where $\gamma \in H_3(\mathbb{C}^2 \times \mathbb{C}^* \backslash Y, \mathbb{Z})$, and the B-brane central charges are defined as certain hypergeometric series in a similar manner to the Gamma series we used here.

A straightforward generalization of this example to higher dimensions is the following.

Definition 60 (Higher-dimensional Hori-Vafa mirrors)**.** The B-models are given by $(d+1)$-dimensional toric Calabi-Yau orbifolds $\mathbb{P}_{\mathbf{\Sigma}}$ as in subsection 2.4.1, where $\mathbf{\Sigma}$ is a triangulation supported on a cone C over a lattice polytope Δ of dimension d and height 1. Its mirror is defined as

$$Y = \{(u, v, \mathbf{w}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^d : u^2 + v^2 + f(\mathbf{w}) = 0\}$$

where f is a Laurent polynomial whose Newton polytope is Δ. The A-brane central charge is defined as

$$\int_\gamma \frac{1}{u^2 + v^2 + f(a; \mathbf{w})} dudv \frac{d\mathbf{w}}{\mathbf{w}}$$

where $\gamma \in H_{d+2}\left(\mathbb{C}^2 \times (\mathbb{C}^*)^d \backslash Y, \mathbb{Z}\right)$.

In this dissertation, we consider a slightly different version of the generalization to higher dimensions. Instead of a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^d$, we simply consider the hypersurface defined by the vanishing of the Laurent polynomial f in the algebraic torus $(\mathbb{C}^*)^d$. Furthermore, we define our version of A-brane central charges as period integrals over certain cycles in the homology group $H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$.

Remark 61. We note that there are some issues with this version of Hori-Vafa mirrors. The first issue is that its dimension does not match with that of the toric stack $\mathbb{P}_{\mathbf{\Sigma}}$. The second issue is that the homology group $H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$ is not quite the correct space of A-branes since its dimension is equal to $\mathrm{Vol}(\Delta) + d$ (while the correct dimension should be $\mathrm{Vol}(\Delta)$), see [3, Proposition 5.2]. We nevertheless stick to this definition because it is more aligned with the setting of [3], and the computation of the asymptotics of the period integral is easier. The main result of this chapter shows that the integral structure on the (slightly "wrong") homology group $H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$ matches with the integral structure on $K_0(\mathbb{P}_\Sigma)$.

We briefly summarize our version of Hori-Vafa mirrors in the following table.

	A-side	B-side
spaces	Laurent polynomial $f : (\mathbb{C}^*)^d \to \mathbb{C}$	toric CY orbifolds $\mathbb{P}_\Sigma$
(A- or B-)branes	certain Lagrangian submanifolds L of $(\mathbb{C}^*)^d$	coherent sheaves on $\mathbb{P}_\Sigma$
spaces of branes	$H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$	$K_0(\mathbb{P}_\Sigma, \mathbb{Z})$
central charges	period integrals	hypergeometric series

Table 4.1: The mirror symmetry setting

The second main result of this dissertation is the following. See Remark 5 for a more detailed explanation of the motivation behind this result.

Theorem 62. *The A- and B-model integral structures of the Hori-Vafa mirrors, defined by* $H_d\left((\mathbb{C}^*)^d \backslash Z_f, \mathbb{Z}\right)$ *and* $K_0(\mathbb{P}_\Sigma, \mathbb{Z})$ *respectively, coincide.*

This chapter is organized as follows. In Section 4.1 we give the definitions of our (modified) version of central charges, in terms of period integrals and Gamma series respectively. In Section 4.3 we prove a technical result on the relationship between certain integral of orbifold cohomology classes on toric stacks and the volume of certain polytopes, which is essential to the computation in Section 4.2. Finally in Section 4.4 we establish the desired equality between A- and B-brane central charges.

4.1 Central charges as solutions to better-behaved GKZ systems

In this section, we give precise definitions of the A-brane and B-brane central charges for Hori-Vafa mirror pairs. Our definitions differ slightly from the ones in [19]. Along the way we briefly recall the basic preliminaries needed to understand the results and arguments.

4.1.1 A-brane central charges

First we fix notations that will be used throughout this chapter. Denote the coordinates on the torus $(\mathbb{C}^*)^d$ by $z = (z_1, \cdots, z_d)$. Consider the Laurent polynomial $f = \sum_{i=1}^n x_i z^{\bar{v}_i}$. We denote by $\bar{v}_i$ the d-dimensional vector obtained from v_i by deleting the last coordinate 1. Similarly, for a lattice point $c \in N_{\mathbb{R}}$, we write $c = (\bar{c}, \deg c)$ where $\deg c$ is the last coordinate of c and $\bar{c}$ consists of the first d coordinates.

Definition 63. For each lattice point c in the interior C° of the cone C, we define the following holomorphic form

$$\omega_c := (-1)^{\deg c - 1}(\deg c - 1)! \frac{z^{\bar{c}}}{f^{\deg c}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}$$

on the complement $(\mathbb{C}^*)^d \backslash Z_f$, where $f = \sum x_i z^{\bar{v}_i}$. The *A-brane central charge* associated to a Lagrangian submanifold L (with certain admissibility conditions) is defined to be the collection of period integrals[1]

$$Z^{A,L}(x) = (Z_c^{A,L}(x))_{c \in C^\circ} = \left(\frac{(-1)^d}{(2\pi \mathrm{i})^{d+1}} \int_L \omega_c \right)_{c \in C^\circ}$$

where each $\int_L \omega_c$ is viewed as a holomorphic function with the coefficients x_i of f as variables.

Since we are mostly interested in the mirror cycles of line bundles in this paper, from now on we will assume L to be the Lagrangian sections of the fibration $\pi : (\mathbb{C}^*)^d \to \mathbb{R}^d$ defined by $z \mapsto \log |z|$. The following result explains the reason why it is more natural to think of the period integrals over Lagrangian sections as solutions to

[1] Note that the constant factor $\frac{(-1)^d}{(2\pi \mathrm{i})^{d+1}}$ plays no essential role.

the dual system bbGKZ$(C^\circ, 0)$ rather than the usual system bbGKZ$(C, 0)$. Similar results can be found in [27] and [4].

Lemma 64. The period integral

$$\int_{\mathbb{R}^d_{\geq 0}} \frac{z^{\bar{c}}}{f(z)^{\deg c}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}$$

is absolutely convergent if and only if $c \in C^\circ$.

Proof. We make the coordinate change $z_i = e^{y_i}$, then the integral becomes

$$\int_{\mathbb{R}^d} \frac{e^{\bar{c}\cdot y}}{f(e^y)^{\deg c}} dy_1 \wedge \cdots \wedge dy_d = \int_{\mathbb{R}^d} \frac{e^{\bar{c}\cdot y}}{(\sum_{j\in\Delta} x_j e^{\bar{v_j}\cdot y})^{\deg c}} dy_1 \wedge \cdots \wedge dy_d$$

Now we divide the space $\mathbb{R}^d$ into cone regions according to the normal fan Σ of the polytope Δ. More precisely, we divide $\mathbb{R}^d$ as the union of the following cone regions

$$\{\sigma_{\bar{v_k}} := -\left(\mathbb{R}_{\geq 0}(\Delta - \bar{v_k})\right)^\vee : \bar{v_k} \in \Delta\}$$

note that this differs from the usual definition of the normal fan by a negative sign. Fix a cone region $\sigma_{\bar{v_k}}$, it's easy to see that over $\sigma_{\bar{v_k}}$ the dominant term in the denominator $\sum_{j\in\Delta} x_j e^{\bar{v_j}\cdot y}$ is exactly the monomial $x_k e^{\bar{v_k}\cdot y}$. Therefore it suffices to consider the absolute convergence of

$$\int_{\sigma_{v_k}} e^{(\bar{c}-(\deg c)\bar{v_k})\cdot y} dy_1 \wedge \cdots \wedge dy_d$$

which is again equivalent to the condition that

$$(\bar{c} - (\deg c)\bar{v_k}) \cdot y < 0, \quad \forall \text{ ray generators } y \text{ of } \sigma_{\bar{v_k}} \text{ and } \forall \bar{v_k} \in \Delta \tag{4.1.1}$$

Recall that the polytope Δ could be defined as the intersection of half-spaces (i.e., the facet representation of Δ):

$$\Delta = \bigcap_{F \text{ facet of } \Delta} (h_F \geq 0)$$

where h_F is the defining equation of the support hyperplane spanned by the facet F. It is then straightforward to observe that the condition (4.1.1) is equivalent to that $\bar{c}/(\deg c)$ is an interior point of the polytope Δ, which is again equivalent to that c is in the interior of the cone C. □

Now we consider a general Lagrangian section L. Additional restrictions are required to ensure the absolute convergence of the period integral. We write the section $L : \mathbb{R}^d \to (\mathbb{C}^*)^d$ as

$$(y_1, \cdots, y_d) \mapsto (e^{y_1} \cdot e^{\mathrm{i}\theta_1(y_1,\cdots,y_d)}, \cdots, e^{y_d} \cdot e^{\mathrm{i}\theta_d(y_1,\cdots,y_d)})$$

Proposition 65. Suppose $L : \mathbb{R}^d \to (\mathbb{C}^*)^d$ is a section of the fibration $T \to \mathbb{R}^d$ such that $\det(I_d + \left(\frac{\partial \theta_i}{\partial y_j}\right)_{i,j})$ is bounded, then the integral $\int_L \omega_c$ is absolutely convergent for all $c \in C^\circ$.

Proof. Follows directly from the last lemma and the observation that $\det(I_d + \left(\frac{\partial \theta_i}{\partial y_j}\right)_{i,j})$ is the determinant of the Jacobian of the change of variables. □

Proposition 66. Suppose γ satisfies the condition in Proposition 65, then $\Psi = (\Psi_c)_{c \in C^\circ}$ where

$$\Psi_c(x_1, \cdots, x_n) := \int_L \omega_c$$

gives a solution to the system bbGKZ$(C^\circ, 0)$.

Proof. The idea of the proof comes from Batyrev [3] and Borisov-Horja [7]. However, since the cycles we are integrating over are non-compact, some additional care must be taken.

To prove the equation $\partial_i \Psi_c = \Psi_{c+v_i}$ for any i, note that we have

$$\partial_i \left(\frac{z^{\bar{c}}}{f(z)^{\deg c}} \right) = (-\deg c) \frac{z^{\bar{c}+\bar{v}_i}}{f(z)^{\deg(c+v_i)}}$$

which gives $\partial_i \omega_c = \omega_{c+v_i}$, and the absolute convergence of the integral ensures that differentiation commutes with integration.

To prove the second equation

$$\sum_{i=1}^{n} \langle \mu, v_i \rangle x_i \partial_i \Psi_c + \langle \mu, c \rangle \Psi_c = 0, \quad \forall \mu \in N^\vee$$

we look at the standard basis $\mu_1, \cdots, \mu_{d+1}$ of $N^\vee$. For $1 \leq k \leq d$, an elementary computation shows that $\sum_{i=1}^{n} \langle \mu_k, v_i \rangle x_i \omega_{c+v_i} + \langle \mu_k, c \rangle \omega_c$ is equal to

$$(-1)^{\deg c - 1} (\deg c - 1)! z_k \partial_{z_k} \left(\frac{z^{\bar{c}}}{f^{\deg c}} \right) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}$$

Note that

$$z_k \partial_{z_k} \left(\frac{z^{\bar{c}}}{f^{\deg c}} \right) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d} = d \left(\frac{z^{\bar{c}}}{f^{\deg c}} \frac{dz_1}{z_1} \wedge \cdots \wedge \widehat{\frac{dz_k}{z_k}} \wedge \cdots \frac{dz_d}{z_d} \right)$$

Take a chain of compact subsets $B_1 \subseteq B_2 \subseteq \cdots \subseteq T$ such that $\cup B_m = T$ (e.g., take B_m to be the box defined by $e^{-m} \leq |z_j| \leq e^m$). By Stokes' theorem the integration of $z_k \partial_{z_k} \left(\frac{z^{\bar{c}}}{f^{\deg c}} \right)$ over $B_i \cap L$ is equal to

$$\int_{\partial(B_m \cap L)} \frac{z^{\bar{c}}}{f^{\deg c}} \frac{dz_1}{z_1} \wedge \cdots \wedge \widehat{\frac{dz_k}{z_k}} \wedge \cdots \frac{dz_d}{z_d}$$

which tends to 0 when $m \to +\infty$ due to the absolute convergence of

$$\int_L \frac{z^{\bar{c}}}{f^{\deg c}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}.$$

On the other hand, by dominated convergence theorem the sequence of integrals converges to the integration of $z_k \partial_{z_k} \left(\frac{z^{\bar{c}}}{f^{\deg c}} \right)$ over L. This finishes the proof of the case when $1 \leq k \leq d$. Finally, if $k = d+1$, i.e., $\mu_k = \deg$, then an elementary computation shows that $\sum_{i=1}^{n} \langle \mu_k, v_i \rangle x_i \omega_{c+v_i} + \langle \mu_k, c \rangle \omega_c$ is zero. □

4.1.2 B-brane central charges

Now we define central charges on the B-brane, i.e., the toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$, in terms of certain cohomology-valued Gamma series..

Recall from Section 3.1 We have define the cohomology-valued Gamma series Γ and Γ° as

$$\Gamma_c(x_1, \ldots, x_n) = \bigoplus_{\gamma} \sum_{l \in L_{c,\gamma}} \prod_{i=1}^{n} \frac{x_i^{l_i + \frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1 + l_i + \frac{D_i}{2\pi \mathrm{i}})} \tag{4.1.2}$$

for lattice point $c \in C$ and

$$\Gamma_c^\circ(x_1, \ldots, x_n) = \bigoplus_{\gamma} \sum_{l \in L_{c,\gamma}} \prod_{i=1}^{n} \frac{x_i^{l_i + \frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1 + l_i + \frac{D_i}{2\pi \mathrm{i}})} \left(\prod_{i \in \sigma} D_i^{-1} \right) F_\sigma$$

for lattice point $c \in C^\circ$, where both direct sums are taken over twisted sectors $\gamma = \sum_{j \in \sigma(\gamma)} \gamma_j v_j$ and the set $L_{c,\gamma}$ is the set of solutions to $\sum_{i=1}^{n} l_i v_i = -c$ with $l_i - \gamma_i \in \mathbb{Z}$ for all i, and σ is the set of i with $l_i \in \mathbb{Z}_{<0}$. The numerator is defined by picking a branch of $\log(x_i)$. It is proved in [5] that these series converge absolutely and uniformly on compacts in a neighborhood of the large radius limit point corresponding to the triangulation Σ. After composing them with linear functions on the orbifold cohomology spaces, we get holomorphic functions with values in $\mathbb{C}$. It is proved that all solutions to the systems $\mathrm{bbGKZ}(C, 0)$ and $\mathrm{bbGKZ}(C^\circ, 0)$ are obtained by composing Γ and Γ° with linear functions on $H^*_{\mathrm{orb}}(\mathbb{P}_{\mathbf{\Sigma}})$ and $H^*_{\mathrm{orb},c}(\mathbb{P}_{\mathbf{\Sigma}})$.

We define the B-brane central charge as follows.

Definition 67. Let N, C, and $\mathbf{\Sigma} = (\Sigma, \{v_i\})$ be as previous. For any class $\mathcal{E} \in K_0(\mathbb{P}_{\mathbf{\Sigma}})$ in the K-theory of $\mathbb{P}_{\mathbf{\Sigma}}$, we define its *B-brane central charge* by

$$Z^{B,\mathcal{E}}(x) = (Z_c^{B,\mathcal{E}}(x))_{c\in C^\circ} = (\chi(ch(\mathcal{E}), -) \circ \Gamma_c^\circ)_{c\in C^\circ}.$$

4.2 Asymptotic behavior of period integrals via tropical geometry

The goal of this section is to analyze the asymptotic behavior of the A-brane central charge associated to the real positive locus $(\mathbb{R}_{>0})^d$.

To begin, we set up the notations that will be used throughout this section. We will be using the same notations as in §4.1. Additionally, let Σ be a regular triangulation of the cone C, and denote the corresponding convex piecewise linear function by ψ. For each lattice point c in the interior C°, we denote the minimal cone in Σ containing c by $\sigma(c)$, and write $c = \sum_{i\in\sigma(c)} c_i v_i$. Then there is a unique twisted sector $\gamma(c) \in \mathrm{Box}(\Sigma)$ given by $\sum_{i\in\sigma(c)}\{c_i\}v_i$, where $\{c_i\}$ denotes the fractional part of c_i. Finally, we denote the set of indices i such that $c_i = 1$ by I_c. Note that $\sigma(c)$ is a disjoint union of I_c and $\sigma(\gamma(c))$.

The main result of this section is an asymptotic formula for the A-brane central charges of the positive real Lagrangian $\mathbb{R}^d_{>0}$ when the parameter approaches the large radius limit corresponding to the triangulation Σ, which should correspond to the structure sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$ on the B-model according to the prediction of SYZ conjecture.

For simplicity, we introduce an extra variable $t \in \mathbb{R}$ and consider the one-parameter family of Laurent polynomials $\{f_t\}$ where $f_t = \sum t^{-\psi(v_i)} z^{v_i}$. Then the large radius limit is achieved by taking $t \to +\infty$. Moreover, to emphasize the dependence of the form ω_c on the parameter t, we adopt the notation $\omega_{t,c}$ instead.

Theorem 68. *The asymptotic behavior of*

$$Z_c^{A,\mathbb{R}^d_{>0}}(t^{-\psi(v_1)},\ldots) = \frac{(-1)^d}{(2\pi\mathrm{i})^{d+1}}\int_{\mathbb{R}^d_{>0}} \omega_{t,c}$$

when $t \to +\infty$ is given by

$$Z_c^{A,\mathbb{R}^d_{>0}}(t^{-\psi(v_1)},\ldots) = t^{\psi(c)}\frac{(-1)^{\mathrm{rk}N-\deg c}}{(2\pi\mathrm{i})^{|\sigma(c)|}|\operatorname{Box}(\sigma(\gamma))|}\cdot\int_{\gamma(c)} t^{\omega}\cdot\hat{\Gamma}_{\gamma(c)}F_{I_c} + o(t^{\psi(c)})$$

where $\omega = \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^n \psi(v_i)D_i$, and $\hat{\Gamma}_{\gamma(c)}$ is the Gamma class of $\gamma(c)$ as defined in Definition 32.

The proof of this theorem occupies the rest of this section. To begin with, we make a change of coordinates. We denote the moment map on $(\mathbb{C}^*)^d$ by

$$\mathrm{Log}_t : (\mathbb{C}^*)^d \to \mathbb{R}^d, (z_1,\cdots,z_d) \mapsto (\log_t|z_1|,\cdots,\log_t|z_d|)$$

and its right-inverse by

$$i_t : \mathbb{R}^d \to (\mathbb{C}^*)^d, (y_1,\cdots,y_d) \mapsto (t^{y_1},\cdots,t^{y_d}).$$

Note that the positive real locus is identified with $\mathbb{R}^d$ under the map i_t. The original integration now becomes

$$\int_{\mathbb{R}^d_{>0}} \omega_{t,c} = \int_{\mathbb{R}^d} i_t^*\omega_{t,c}$$

with the new coordinates $\{y_i\}$.

We divide the proof into two steps. In the first step (§4.2.1), we partition the domain $\mathbb{R}^d$ into smaller sections based on the tropicalization of the Laurent polynomial f_t. This allows us to establish a connection between the integration over each section and the volume of specific polytopes. Moving onto the second step (§4.2.2) we establish a relationship between these integrals and the integral of Gamma classes on the toric stack $\mathbb{P}_{\mathbf{\Sigma}}$. A crucial component of the second step is a Duistermaat-Heckman type lemma adapted to our setting that is stated and proved in Appendix 4.3.

4.2.1 Subdivision of the domain

For each $i \in \Delta$, we consider the tropicalization β_i of the monomial $t^{-\psi(v_i)}z^{v_i}$ defined as

$$\beta_i : \mathbb{R}^d \to \mathbb{R}, \quad p \mapsto \langle v_i, p\rangle - \psi(v_i)$$

which is an affine function on $\mathbb{R}^d$. Following the idea of [1], we define[2]

$$U^q := \{p \in \mathbb{R}^d : \beta_i(p) \leq \beta_q(p), \forall i \in \Delta\}$$

for any lattice point $q \in \Delta$. Furthermore, for any $q \in \Delta$ and any subset $q \notin K \subseteq \Delta$ we define a subset $U^{q,K}$ of U^q by

$$U^{q,K} := \{p \in \mathbb{R}^d : \beta_q(p) - \beta_i(p) \in [0, \epsilon], \forall i \in K, \beta_q(p) - \beta_i(p) \in [\epsilon, +\infty), \forall i \notin q \sqcup K\}$$

for some fixed small positive number $\epsilon > 0$. Intuitively, $U^{q,K}$ is the region where the tropical monomial β_q is the largest (hence dominates the asymptotics) and the tropical monomials $\{\beta_k\}_{k\in K}$ are not far behind.

Remark 69. By the standard argument of tropical geometry, we have the following facts about U^q and $U^{q,K}$. Firstly, the set $U^{q,K}$ is non-empty if and only if $q \sqcup K$ forms a cone in Σ. Additionally, $U^{q,K}$ is unbounded if and only if $q \sqcup K$ is a cone on the boundary, i.e., $\mathrm{relint}(q \sqcup K) \subseteq \partial\Delta$. We will not use the latter fact in rest of the paper

[2] Our definition differs from the one in [1] by reversing the direction of the inequality, due to the difference between $t \to +\infty$ and $t \to 0^+$.

so we omit its proof.

Hence the original integral can be written as a sum

$$\int_{\mathbb{R}^d} i_t^* \omega_{t,c} = \sum_{q,K} \int_{U^{q,K}} i_t^* \omega_{t,c}$$

The first observation is the following lemma which states that only the region $U^{q,K}$ with $\sigma(c) \subseteq q \sqcup K \in \Sigma$ contributes to the leading term when $t \to +\infty$. Otherwise the growth of the integral over the piece will be $O(t^{\psi(c)-\epsilon})$ for some $\epsilon > 0$.

Lemma 70. As $t \to +\infty$, for q and K with $\sigma(c) \not\subseteq q \sqcup K$ we have $\int_{U^{q,K}} i_t^* \omega_{t,c} = O(t^{\psi(c)-\epsilon})$ for some $\epsilon > 0$. If $\sigma(c) \subseteq q \sqcup K \in \Sigma$, then $\int_{U^{q,K}} i_t^* \omega_{t,c}$ is

$$(-1)^{\deg c - 1}(\deg c - 1)! t^{\psi(c)} (\log t)^d \int_{U^{q,K}} \frac{\prod_{i \in K} (t^{\beta_i - \beta_q})^{c_i}}{\left(1 + \sum_{i \in K} t^{\beta_i - \beta_q}\right)^{\deg c}} \prod_i dy_i$$
$$+ O(t^{\psi(c)-\epsilon}(\log t)^d).$$

Proof. First we suppose $\sigma(c) \not\subseteq q \sqcup K$. We have

$$\begin{aligned}
z^{\bar{c}} = z^{\sum_{i \in \sigma(c)} c_i \bar{v}_i} &= z^{c_q \bar{v}_q} \cdot \prod_{i \in \sigma(c) \cup K \setminus q} z^{c_i \bar{v}_i} \\
&= t^{\sum_{i \in \sigma(c)} c_i \psi(v_i)} (t^{-\psi(v_q)} z^{\bar{v_q}})^{c_q} \cdot \prod_{i \in \sigma(c) \cup K \setminus q} (t^{-\psi(v_i)} z^{\bar{v_i}})^{c_i} \\
&= t^{\sum_{i \in \sigma(c)} c_i \psi(v_i)} (t^{-\psi(v_q)} z^{\bar{v_q}})^{\sum_{i \in q \sqcup K \cup \sigma(c)} c_i} \cdot \prod_{i \in \sigma(c) \cup K \setminus q} (t^{-\psi(v_i) + \psi(v_q)} z^{\bar{v_i} - \bar{v_q}})^{c_i} \\
&= t^{\psi(c)} (t^{-\psi(v_q)} z^{\bar{v_q}})^{\deg c} \cdot \prod_{i \in \sigma(c) \cup K \setminus q} (t^{-\psi(v_i) + \psi(v_q)} z^{\bar{v_i} - \bar{v_q}})^{c_i}
\end{aligned}$$

Similarly we can compute

$$f_t(z)^{\deg c} = \Big(\sum_{i \in \Delta} t^{-\psi(v_i)} z^{\bar{v_i}}\Big)^{\deg c}$$

$$= (t^{-\psi(v_q)} z^{\bar{v_q}})^{\deg c} \cdot (1 + \sum_{i \in K} t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}} + \sum_{j \notin q \sqcup K} t^{-\psi(v_j)+\psi(v_q)} z^{\bar{v_j}-\bar{v_q}})^{\deg c}$$

$$= (t^{-\psi(v_q)} z^{\bar{v_q}})^{\deg c} \cdot \left(1 + \sum_{i \in K} t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}} + O(t^{-\epsilon})\right)^{\deg c}$$

Hence the form $\omega_{t,c}$ is

$$\omega_{t,c} = (-1)^{\deg c - 1} (\deg c - 1)! t^{\psi(c)} \frac{\prod_{i \in \sigma(c) \cup K \setminus q} (t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}})^{c_i}}{\left(1 + \sum_{i \in K} t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}} + O(t^{-\epsilon})\right)^{\deg c}} \cdot \prod_i \frac{dz_i}{z_i}$$

$$= (-1)^{\deg c - 1} (\deg c - 1)! t^{\psi(c)} \left(\frac{\prod_{i \in \sigma(c) \cup K \setminus q} (t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}})^{c_i}}{\left(1 + \sum_{i \in K} t^{-\psi(v_i)+\psi(v_q)} z^{\bar{v_i}-\bar{v_q}}\right)^{\deg c}} + O(t^{-\epsilon}) \right) \cdot \prod_i \frac{dz_i}{z_i}$$

Therefore the pullback $i_t^* \omega_{t,c}$ is

$$i_t^* \omega_{t,c} = (-1)^{\deg c - 1} (\deg c - 1)! t^{\psi(c)} \left(\frac{\prod_{i \in \sigma(c) \cup K \setminus q} (t^{\beta_i - \beta_q})^{c_i}}{\left(1 + \sum_{i \in K} t^{\beta_i - \beta_q}\right)^{\deg c}} + O(t^{-\epsilon}) \right) \cdot (\log t)^d \prod_i dy_i$$

So the integration $\int_{U^{q,K}} i_t^* \omega_{t,c}$ is equal to

$$(-1)^{\deg c - 1} (\deg c - 1)! t^{\psi(c)} (\log t)^d \int_{U^{q,K}} \frac{\prod_{i \in \sigma(c) \cup K \setminus q} (t^{\beta_i - \beta_q})^{c_i}}{\left(1 + \sum_{i \in K} t^{\beta_i - \beta_q}\right)^{\deg c}} \prod_i dy_i$$
$$+ O(t^{\psi(c) - \epsilon} (\log t)^d).$$

Notice that if $\sigma(c) \not\subseteq q \sqcup K$, then there exists $i \in \sigma(c)$ such that $\beta_q - \beta_i \geq \epsilon$, where $\epsilon > 0$ is the constant used in the definition of $U^{q,K}$, i.e., the nominator of the integrand will contribute a factor of $t^{-\epsilon}$. Therefore the first term in the above expression is also $O(t^{\psi(c)-\epsilon} (\log t)^d)$. By changing ϵ to a smaller positive number we can adsorb the logarithmic term and get $O(t^{\psi(c)-\epsilon})$.

Now suppose $\sigma(c) \subseteq q \sqcup K$. Then the same computation as above shows that the integral $\int_{U^{q,K}} i_t^* \omega_{t,c}$ is equal to

$$(-1)^{\deg c - 1} (\deg c - 1)! t^{\psi(c)} (\log t)^d \int_{U^{q,K}} \frac{\prod_{i \in K} (t^{\beta_i - \beta_q})^{c_i}}{\left(1 + \sum_{i \in K} t^{\beta_i - \beta_q}\right)^{\deg c}} \prod_i dy_i$$

$$+ O(t^{\psi(c)-\epsilon}(\log t)^d).$$

□

According to this lemma we can disregard integrals over $U^{q,K}$ with $\sigma(c) \not\subseteq q \sqcup K$ when computing the leading term of the asymptotic behavior.

We consider the integral

$$\int_{U^{q,K}} \frac{\prod_{i\in K}(t^{\beta_i-\beta_q})^{c_i}}{\left(1+\sum_{i\in K} t^{\beta_i-\beta_q}\right)^{\deg c}} \prod_i dy_i \tag{4.2.1}$$

where (q, K) is a fixed pair with $\sigma(c) \subseteq q \sqcup K$. To simplify the expression, we introduce a change of coordinate on the region $U^{q,K}$. Let us define

$$b_i := \beta_q - \beta_i, \text{ for all } i \in K$$

and complete $\{b_i\}_{i\in K}$ into an affine coordinate system on $\mathbb{R}^d$ by adding additional covectors $\{e_j\}$. We can then express the standard affine volume form on $\mathbb{R}^d$ in terms of this new system of coordinates:

$$\prod_i dy_i = r_{q,K} \cdot \prod_{i\in K} db_i \cdot \prod_j de_j.$$

Thus, the original integral becomes

$$\int_{U^{q,K}} \frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}} \cdot r_{q,K} \cdot \prod_{i\in K} db_i \cdot \prod_j de_j.$$

Recall that the region $U^{q,K}$ is defined as

$$U^{q,K} = \left\{p \in \mathbb{R}^d : b_i \in [0, \epsilon], \forall i \in K;\ \beta_q - \beta_i \in [\epsilon, \infty], \forall i \notin q \sqcup K\right\}$$

We consider the projection $\pi_b : U^{q,K} \to [0,\epsilon]^K$ onto the $(b_i)_{i\in K}$-coordinate plane, and denote the fiber of a fixed $(b_i)_{i\in K} \in [0,\epsilon]^K$ by $F^{q,K}((b_i)_{i\in K})$. Then the integral above can be written as an iterated integral

$$\begin{aligned}
&\int_{U^{q,K}} \frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}} \cdot r_{q,K} \cdot \prod_{i\in K} db_i \cdot \prod_j de_j \\
=&\int_{[0,\epsilon]^K} \left(\int_{F^{q,K}((b_i)_{i\in K})} \frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}} \cdot r_{q,K} \cdot \prod_j de_j \right) \prod_{i\in K} db_i \\
=&\int_{[0,\epsilon]^K} \left(\frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}} \cdot \operatorname{vol}\left(F^{q,K}((b_i)_{i\in K})\right) \right) \prod_{i\in K} db_i.
\end{aligned}$$

By definition, the fiber $F^{q,K}((b_i)_{i\in K})$ is given by

$$F^{q,K}((b_i)_{i\in K}) = \left\{ p \in \mathbb{R}^d : \beta_q - \beta_i = b_i, \forall i \in K;\ \beta_q - \beta_i \in [\epsilon, \infty], \forall i \notin q \sqcup K \right\}.$$

To remove the ϵ-dependence, we introduce a new polytope

$$E^{q,K}((b_i)_{i\in K}) = \left\{ p \in \mathbb{R}^d : \beta_q - \beta_i = b_i, \forall i \in K;\ \beta_q - \beta_i \in [0, \infty], \forall i \notin q \sqcup K \right\}.$$

and express the fiber $F^{q,K}((b_i)_{i\in K})$ in terms of these new polytopes:

$$F^{q,K}((b_i)_{i\in K}) = E^{q,K}((b_i)_{i\in K}) \setminus \left(\bigcup_{j\notin q\sqcup K} \bigcup_{b_j \in [0,\epsilon]} E^{q,K\sqcup\{j\}}((b_i)_{i\in K}, b_j) \right).$$

By inclusion-exclusion principle, we have

$$\operatorname{vol}(F^{q,K}((b_i)_{i\in K})) = \sum_{J: J\supseteq K, q\notin J} (-1)^{|J\setminus K|} \int_{[0,\epsilon]^{J\setminus K}} \operatorname{vol}(E^{q,J}((b_i)_{i\in K}, (b'_j)_{j\in J\setminus K}))db'.$$

Combining these results, the integral (4.2.1) becomes

$$\int_{[0,\epsilon]^K} \frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}}.$$

$$\left(\sum_{J:J\supseteq K, q\notin J} (-1)^{|J\setminus K|} \int_{[0,\epsilon]^{J\setminus K}} \mathrm{vol}(E^{q,J}((b_i)_{i\in K}, (b'_j)_{j\in J\setminus K}))db' \right) db.$$

By allowing q and K to vary, we obtain

$$\sum_{(J,K,q):J\supseteq K, q\notin J} (-1)^{|J\setminus K|} \int_{[0,\epsilon]^J} \frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\deg c}} \cdot \mathrm{vol}(E^{q,J}((b_i)_{i\in K}, (b'_j)_{j\in J\setminus K}))db' db, \tag{4.2.2}$$

where the sum is taken over all triples (J, K, q) such that $K \subseteq J$ and $q \notin J$. Note that the summand corresponding to J is zero unless $q \sqcup J$ is a cone in Σ.

4.2.2 Connection to Gamma classes

The goal of this subsection is to reveal the relationship between (4.2.2) with the Gamma classes Γ_γ of twisted sectors of the toric stack $\mathbb{P}_{\mathbf{\Sigma}}$. We adopt a similar approach as presented in [1].

To begin, we apply the following volume formula that relates the volume of the polytope with certain integrals of orbifold cohomology classes on the toric stack $\mathbb{P}_{\mathbf{\Sigma}}$.

Proposition 71. The residual volume $\mathrm{vol}(E^{q,J}((b_j)_{j\in J}))$ is equal to

$$\frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_{\mathbb{P}_{\mathbf{\Sigma}/\gamma}} e^{D-\sum_{j\in J} b_j D_j} \frac{D_{q\sqcup J}}{D_{\sigma(c)}} F_{I_c}$$

where $D := \sum_i \psi(v_i) D_i$, and $\gamma := \gamma(c)$ is the unique twisted sector corresponding to the interior lattice point $c \in C^\circ$. Note that we denote a twisted sector corresponding to $\gamma \in \operatorname{Box}(\Sigma)$ by $\mathbb{P}_{\mathbf{\Sigma}/\gamma}$ to avoid potential confusion.

The proof of this formula will be postponed to the next section.

The sum (4.2.2) obtained in the previous subsection can be rewritten as

$$\begin{aligned}\sum_{(J,K,q):J\supseteq K,q\notin J}(-1)^{|J\setminus K|}\int_{[0,\epsilon]^J}\frac{t^{-\sum_{i\in K}c_ib_i}}{\left(1+\sum_{i\in K}t^{-b_i}\right)^{\deg c}}\cdot \\ \left(\frac{1}{|\operatorname{Box}(\sigma(\gamma))|}\int_\gamma e^{D-\sum_{j\in J}b_jD_j}\cdot\frac{D_{q\sqcup J}}{D_{\sigma(d)}}\cdot F_{I_c}\right)db'db\end{aligned} \tag{4.2.3}$$

where $D=\sum\psi(v_i)D_i$ and $\gamma:=\gamma(c)$ is the unique twisted sector corresponding to the lattice point $c\in C^\circ$. Now we consider the following cohomology class in H^*_γ obtained by scaling all classes D_i by a factor of $\frac{\log t}{2\pi\mathrm{i}}$:

$$\begin{aligned}P_t=\sum_{(J,K,q):J\supseteq K,q\notin J}(-1)^{|J\setminus K|}\left(\frac{\log t}{2\pi\mathrm{i}}\right)^{|J|+1-|\sigma(c)|}\frac{D_{q\sqcup J}}{D_{\sigma(c)}} \\ \cdot\int_{[0,\epsilon]^J}\left(\frac{t^{-\sum_{i\in K}c_ib_i}}{\left(1+\sum_{i\in K}t^{-b_i}\right)^{\deg c}}e^{(\log t)\frac{D}{2\pi\mathrm{i}}-\sum_{j\in J}b_j(\log t)\frac{D_j}{2\pi\mathrm{i}}}\right)db'db\end{aligned}$$

Since the integral over γ is only relevant to the deg $=\dim\gamma=d+1-|\sigma(\gamma)|$ part, the expression (4.2.3) is equal to

$$\frac{1}{|\operatorname{Box}(\sigma(\gamma))|}\left(\frac{\log t}{2\pi\mathrm{i}}\right)^{-(d+1-|\sigma(c)|)}\int_\gamma P_t\cdot F_{I_c} \tag{4.2.4}$$

Note that $\deg F_{I_c}=|I_c|$ and $|\sigma(\gamma)|+|I_c|=|\sigma(c)|$.

The goal of the remaining part of this subsection is to prove the following result which relates the cohomology class P_t to the Gamma class $\hat{\Gamma}_\gamma$ of the twisted sector γ.

Proposition 72. The asymptotics of the class P_t is given by

$$P_t=\frac{(\log t)^{1-|\sigma(c)|}}{(\deg c-1)!}t^\omega\widehat{\Gamma}_\gamma+O(t^{-\epsilon})$$

where $\omega=\frac{1}{2\pi\mathrm{i}}\sum\psi(v_i)D_i$.

To proceed, we consider the following analytic function in $D_1,\cdots,D_n$, where we think of the variables D_i's as usual complex numbers:

$$Q_t(D_1, \cdots, D_n) = \sum_{(J,K,q):J\supseteq K, q\notin J} (-1)^{|J\setminus K|} \cdot \left(\frac{\log t}{2\pi \mathrm{i}}\right)^{|J|+1-|\sigma(c)|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{D_{q\sqcup J}}{D_{\sigma(c)}}$$
$$\cdot \int_{[0,\epsilon]^J} \left(\frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} b_j (\log t)\frac{D_j}{2\pi \mathrm{i}}} \right) db' db$$

The next proposition establishes the relationship between the function $Q(D_1, \cdots, D_n)$ and the Gamma function Γ.

Proposition 73. As functions in variables D_i's we have the following identity:

$$Q(D_1, \cdots, D_n) = (\log t)^{1-|\sigma(c)|} (2\pi \mathrm{i})^{|\sigma(c)|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{\prod_{i=1}^n \frac{D_i}{2\pi \mathrm{i}}}{D_{\sigma(c)}} \cdot \frac{\prod_{i=1}^n \Gamma(\frac{D_i}{2\pi \mathrm{i}}+c_i)}{\Gamma(\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i))} + O(t^{-\epsilon})$$

Proof. First, we consider a single integral in the definition of Q:

$$\int_{[0,\epsilon]^J} \left(\frac{t^{-\sum_{i\in K} c_i b_i}}{\left(1+\sum_{i\in K} t^{-b_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} b_j (\log t)\frac{D_j}{2\pi \mathrm{i}}} \right) db' db$$

we introduce the change of variables $s_i := (\log t) b_i$ to rewrite it as

$$\int_{[0,\epsilon \log t]^J} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} s_j \frac{D_j}{2\pi \mathrm{i}}} \right) \frac{ds}{(\log t)^{|J|}}$$

We now claim that we could replace the region $[0, \epsilon \log t]^J$ of the integral by $[0, \infty)^J$ without changing the leading term of the asymptotics. In other words, we have

$$\int_{[0,\epsilon \log t]^J} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} s_j \frac{D_j}{2\pi \mathrm{i}}} \right) ds$$
$$= \int_{[0,\infty)^J} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} s_j \frac{D_j}{2\pi \mathrm{i}}} \right) ds + O(t^{-\epsilon}).$$

To see this, it suffices to observe that the integrand is controlled by

$$e^{-\sum_{i\in K} c_i s_i} \cdot e^{-\sum_{j\in J} s_j \frac{D_j}{2\pi \mathrm{i}}}$$

then the claim follows from the fact that $\int_{\epsilon \log t}^{+\infty} e^{-s} ds = O(t^{-\epsilon})$.

Thus it suffices to look at

$$\sum_{(J,K,q):J\supseteq K, q\notin J} (-1)^{|J\backslash K|} \left(\frac{\log t}{2\pi \mathrm{i}}\right)^{|J|+1-|\sigma(c)|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{D_{q\sqcup J}}{D_{\sigma(c)}} \cdot \int_{[0,\infty)^J} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{j\in J} s_j \frac{D_j}{2\pi \mathrm{i}}} \right) \frac{ds}{(\log t)^{|J|}}$$

Note that $\int_0^\infty e^{-s_j \frac{D_j}{2\pi i}} ds_j = (2\pi \mathrm{i})/D_j$, integrating for all $j \in J\backslash K$, the integral becomes

$$\sum_{(J,K,q):J\supseteq K, q\notin J} (-1)^{|J\backslash K|} \left(\frac{\log t}{2\pi \mathrm{i}}\right)^{|J|+1-|\sigma(c)|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{D_{q\sqcup J}}{D_{\sigma(c)}} \cdot \int_{[0,\infty)^K} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{i\in K} s_i \frac{D_i}{2\pi \mathrm{i}}} \prod_{j\in J\backslash K} \frac{2\pi \mathrm{i}}{D_j} \right) \frac{ds}{(\log t)^{|J|}}$$

which is equal to

$$\sum_{(J,K,q):J\supseteq K, q\notin J} (-1)^{|J\backslash K|} \left(\frac{\log t}{2\pi \mathrm{i}}\right)^{1-|\sigma(c)|} \left(\frac{1}{2\pi \mathrm{i}}\right)^{|K|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{D_{q\sqcup K}}{D_{\sigma(c)}} \cdot \int_{[0,\infty)^K} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{i\in K} s_i \frac{D_i}{2\pi \mathrm{i}}} \right) ds$$

We rewrite the sum as

$$\sum_{(K,q):q\notin K} \left(\sum_{J:J\supseteq K, q\notin J} (-1)^{|J\backslash K|} \right) \left(\frac{\log t}{2\pi \mathrm{i}}\right)^{1-|\sigma(c)|} \left(\frac{1}{2\pi \mathrm{i}}\right)^{|K|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{D_{q\sqcup K}}{D_{\sigma(c)}} \cdot \int_{[0,\infty)^K} \left(\frac{e^{-\sum_{i\in K} c_i s_i}}{\left(1+\sum_{i\in K} e^{-s_i}\right)^{\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}}+c_i)}} e^{-\sum_{i\in K} s_i \frac{D_i}{2\pi \mathrm{i}}} \right) ds$$

Note that for a fixed pair (K,q), the sum

$$\sum_{J:q\notin J, K\subseteq J}(-1)^{|J\setminus K|}=\sum_{P\subseteq\{1,\cdots,N\}\setminus(q\sqcup K)}(-1)^{|P|}$$

is equal to $(1+(-1))^{|\{1,\cdots,n\}\setminus(q\sqcup K)|}=0$ if $\{1,\cdots,n\}\neq q\sqcup K$ and 1 otherwise. Therefore, the only nonzero term in the sum above corresponds to $K=\{1,\cdots,n\}\setminus q$. Consequently, the sum above can be expressed as

$$\sum_{q=1}^{n}\left(\frac{\log t}{2\pi\mathrm{i}}\right)^{1-|\sigma(c)|}\left(\frac{1}{2\pi\mathrm{i}}\right)^{n-1}e^{(\frac{\log t}{2\pi\mathrm{i}})D}\frac{\prod_{i=1}^{n}D_i}{D_{\sigma(c)}}$$
$$\cdot\int_{[0,\infty)^K}\left(\frac{e^{-\sum_{i\in K}c_i s_i}}{\left(1+\sum_{i\in K}e^{-s_i}\right)^{\sum_{i=1}^{n}(\frac{D_i}{2\pi\mathrm{i}}+c_i)}}e^{-\sum_{i\in K}s_i\frac{D_i}{2\pi\mathrm{i}}}\right)ds.$$

Simplifying further, we obtain

$$(\log t)^{1-|\sigma(c)|}(2\pi\mathrm{i})^{|\sigma(c)|}\sum_{q=1}^{n}e^{(\frac{\log t}{2\pi\mathrm{i}})D}\frac{\prod_{i=1}^{n}\frac{D_i}{2\pi\mathrm{i}}}{D_{\sigma(c)}}$$
$$\cdot\int_{[0,\infty)^K}\left(\frac{e^{-\sum_{i\in K}c_i s_i}}{\left(1+\sum_{i\in K}e^{-s_i}\right)^{\sum_{i=1}^{n}(\frac{D_i}{2\pi\mathrm{i}}+c_i)}}e^{-\sum_{i\in K}s_i\frac{D_i}{2\pi\mathrm{i}}}\right)ds.$$

We substitute $t_i=e^{-s_i}$, leading to the following expression

$$(\log t)^{1-|\sigma(c)|}(2\pi\mathrm{i})^{|\sigma(c)|}e^{(\frac{\log t}{2\pi\mathrm{i}})D}\frac{\prod_{i=1}^{n}\frac{D_i}{2\pi\mathrm{i}}}{D_{\sigma(c)}}$$
$$\sum_{q\in\{1,\cdots,n\}}\int_{[0,1]^{n-1}}\frac{\prod_{i\neq q}t_i^{\frac{D_i}{2\pi\mathrm{i}}+c_i}}{\left(1+\sum_{i\neq q}t_i\right)^{\sum_{i=1}^{n}(\frac{D_i}{2\pi\mathrm{i}}+c_i)}}\frac{dt}{t}$$

Now we recall the definition and some basic properties of the multivariate Beta function. The multivariable beta function is defined by

$$B(a_1,\cdots,a_n)=\int_{[0,\infty)^{n-1}}\frac{t_1^{a_1}\cdots t_{n-1}^{a_{n-1}}}{(1+t_1+\cdots+t_{n-1})^{a_1+\cdots+a_n}}dt_1\cdots dt_{n-1}$$

and there is an identity $B(a_1, \cdots, a_n) = \Gamma(a_1)\cdots\Gamma(a_n)/\Gamma(a_1 + \cdots + a_n)$.

Lemma 74. We have the following identity:

$$B(a_1, \cdots, a_n) = \sum_{q=1}^{n} \int_{[0,1]^{n-1}} \frac{\prod_{i \neq q} t_i^{a_i - 1}}{\left(1 + \sum_{i \neq q} t_i\right)^{\sum_{i=1}^{n} a_i}} \prod_{i \neq q} dt_i$$

Proof. We treat the $q = n$ term and the other terms separately. The $q = n$ term could be written as

$$\int_{[0,1]^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{a_i-1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^{\sum_{i=1}^{n} a_i}} \prod_{i=1}^{n-1} dt_i = \sum_{q=1}^{n-1} \int_{A_q} \frac{\prod_{i=1}^{n-1} t_i^{a_i-1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^{\sum_{i=1}^{n} a_i}} \prod_{i=1}^{n-1} dt_i$$

where A_q is the region defined by $0 \leq t_q \leq 1$ and $0 \leq t_j \leq t_q$ for $j \neq q$.

When $q \neq n$, we introduce the change of coordinate given by $t_i \to \frac{t_i}{t_n}$ for $i \neq n$ and $t_n \to \frac{1}{t_n}$, then rename the variable t_n to t_q. An elementary computation shows that the integral becomes

$$\int_{B_q} \frac{\prod_{i=1}^{n-1} t_i^{a_i-1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^{\sum_{i=1}^{n} a_i}} \prod_{i=1}^{n-1} dt_i$$

where B_q is the region defined by $t_q \geq 1$ and $0 \leq t_j \leq t_q$ for $j \neq q$. Therefore the original sum can be written as the integral over the union $\bigcup_{q=1}^{n-1}(A_q \cup B_q)$. Now the result follows from the observation that $A_q \cup B_q$ is the region defined by $t_q \geq 0$ and $0 \leq t_j \leq t_q$ for $j \neq q$, and $\bigcup_{q=1}^{n-1}(A_q \cup B_q)$ is exactly $[0, \infty)^{n-1}$. □

Remark 75. A similar identity was proved in [1] by using integration over certain tropical projective spaces. The proof we provide here is purely elementary.

Finally we apply Lemma 74 to rewrite this expression as

$$(\log t)^{1-|\sigma(c)|}(2\pi \mathrm{i})^{|\sigma(c)|} e^{(\frac{\log t}{2\pi \mathrm{i}})D} \frac{\prod_{i=1}^{n} \frac{D_i}{2\pi \mathrm{i}}}{D_{\sigma(c)}} \cdot \frac{\prod_{i=1}^{n} \Gamma(\frac{D_i}{2\pi \mathrm{i}} + c_i)}{\Gamma(\sum_{i=1}^{n}(\frac{D_i}{2\pi \mathrm{i}} + c_i))}.$$

This concludes the proof of the proposition. $\square$

We have established the desired relationship between the function Q_t and the Gamma functions. Applying them to the cohomology classes D_i along with the following equality

$$\begin{aligned}
\frac{\prod_{i=1}^n \frac{D_i}{2\pi \mathrm{i}}}{D_{\sigma(c)}} \cdot \frac{\prod_{i=1}^n \Gamma(\frac{D_i}{2\pi \mathrm{i}} + c_i)}{\Gamma(\sum_{i=1}^n (\frac{D_i}{2\pi \mathrm{i}} + c_i))} &= \frac{1}{(2\pi \mathrm{i})^{|\sigma(c)|}} \frac{\prod_{i \notin \sigma(c)} \frac{D_i}{2\pi \mathrm{i}}}{(\deg c - 1)!} \prod_{i \notin \sigma(c)} \Gamma(\frac{D_i}{2\pi \mathrm{i}}) \prod_{i \in \sigma(c)} \Gamma(\frac{D_i}{2\pi \mathrm{i}} + c_i) \\
&= \frac{1}{(2\pi \mathrm{i})^{|\sigma(c)|} (\deg c - 1)!} \prod_{i \notin \sigma(\gamma)} \Gamma(1 + \frac{D_i}{2\pi \mathrm{i}}) \prod_{i \in \sigma(\gamma)} \Gamma(\frac{D_i}{2\pi \mathrm{i}} + \gamma_i) \\
&= \frac{1}{(2\pi \mathrm{i})^{|\sigma(c)|} (\deg c - 1)!} \cdot \widehat{\Gamma}_\gamma
\end{aligned}$$

we conclude the proof of Proposition 72.

Hence, the leading term of $\frac{(-1)^d}{(2\pi \mathrm{i})^{d+1}} \int_{\mathbb{R}^d_{>0}} \omega_{t,c}$ is given by (noting that $d + 1 - \deg c = \mathrm{rk} N - \deg c$)

$$t^{\psi(c)} \frac{(-1)^{\mathrm{rk} N - \deg c}}{(2\pi \mathrm{i})^{|\sigma(c)|} |\operatorname{Box}(\sigma(\gamma))|} \int_\gamma t^\omega \widehat{\Gamma}_\gamma \cdot F_{I_c}$$

This concludes the proof of Theorem 68.

4.3 Residual volume and orbifold cohomology

In this section we prove the technical volume formula Proposition 71 that relates the residual volume of the polytopes $E^{q,J}((b_j)_{j \in J})$ (for precise definitions of residual volume and the polytopes, see section 4.2) with certain orbifold cohomology classes with compact support of the toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$. It could be considered as a replacement of the Duistermaat-Heckman lemma used in [1] adapted to our setting. We use the same notations from section 4.2 except we denote a twisted sector corresponding to $\gamma \in \operatorname{Box}(\Sigma)$ by $\mathbb{P}_{\mathbf{\Sigma}/\gamma}$ to avoid potential confusion.

We begin with a review of the well-known results on the relationship between line bundles on toric varieties and the associated polytopes in §4.3.1 and provide the proof of Proposition 71 in §4.3.2.

4.3.1 Line bundles on toric varieties and their associated polytopes

We briefly review the classical correspondence between line bundles on toric varieties and their associated polytopes following [14].

Again, we denote by $\mathbb{P}_\Sigma$ the toric variety corresponding to a fan Σ, and $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on $\mathbb{P}_\Sigma$, where D_ρ's are the torus-invariant divisors, and we denote the primitive generators of the correponding rays in the fan by v_ρ. The associated polytope P_D of the line bundle $\mathcal{O}_{\mathbb{P}_\Sigma}(D)$ is defined as[3]

$$P_D := \{m \in M_{\mathbb{R}} : \langle m, v_\rho \rangle + a_\rho \geq 0,\ \forall \rho\}$$

It is a well-known fact that the dimension of the global section of $\mathcal{O}_{\mathbb{P}_\Sigma}(D)$ is equal to the number of lattice points in the polytope P_D. In fact, we have

$$\Gamma(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$$

In this case the polytope P_D is of full-dimension.

This correspondence could be generalized further to the case where the polytope is not of full-dimension. In this case, the corresponding sheaf is not a line bundle on $\mathbb{P}_\Sigma$, but the restriction of a line bundle to a closed subvariety.

Finally, suppose we have a sheaf of the form $\mathcal{O}_{D'}(D)$ where D' and D are torus-invariant divisors and its associated (non-full-dimensional) polytope P. Suppose further that this sheaf is nef. Then by Demazure vanishing theorem (see [14, Theorem

[3] There is a difference of signs in our definition with the one in [14].

9.2.3]) all higher cohomology of $\mathcal{O}_{D'}(D)$ vanishes. Consequently, we have

$$\chi(\mathbb{P}_\Sigma, \mathcal{O}_{D'}(D)) = \chi(D', \mathcal{O}_{D'}(D)) = \dim H^0(D', \mathcal{O}_{D'}(D)) = |P \cap M|$$

i.e., the Euler characteristic of $\mathcal{O}_{D'}(D)$ is equal to the number of lattice points in the polytope P.

4.3.2 Proof of the volume formula

Before we start, we remark here that it suffices to prove the statement for the case where $q \sqcup J$ is a cone in the fan Σ because otherwise the polytope $E^{q,J}((b_j)_{j\in J})$ is empty and the right hand side of the equality is zero due to the factor $D_{q\sqcup J}$.

We divide the proof into four steps.

Step 1. Recall that the polytope $E^{q,J}((b_j)_{j\in J})$ is defined as

$$E^{q,J}((b_i)_{i\in J}) = \left\{p \in \mathbb{R}^d : \beta_q - \beta_i = b_i, \forall i \in J;\ \beta_q - \beta_i \in [0,\infty], \forall i \notin q \sqcup J\right\}$$

where $\beta_i : \mathbb{R}^d \to \mathbb{R}$ is a linear function defined as $p \mapsto \langle v_i, p\rangle - \psi(v_i)$. Without loss of generality, we assume that all b_i's are rational numbers. The defining equalities and inequalities of $E^{q,J}((b_j)_{j\in J})$ could be rewritten as

$$\langle v_q - v_i, p\rangle + \psi(v_i) - \psi(v_q) - b_i = 0$$

for $i \in J$ and

$$\langle v_q - v_i, p\rangle + \psi(v_i) - \psi(v_q) \geq 0$$

for $i \notin q \sqcup J$. If we denote

$$\begin{aligned} D :&= \sum_{i \in J} (\psi(v_i) - \psi(v_q) - b_i) D_i + \sum_{i \notin q \sqcup J} (\psi(v_i) - \psi(v_q)) D_i \\ &= \sum_{i \in \mathrm{Star}(q)} \psi(v_i) D_i - \sum_{i \in J} b_i D_i \end{aligned}$$

then by the discussion in §4.3.1, we have

$$\chi(\mathbb{P}_{\Sigma/q}, \mathcal{O}_{D_J}(lD)) = |l \cdot E^{q,J}((b_i)_{i \in J}) \cap M|$$

where l is any integer number that makes $l \cdot E^{q,J}((b_i)_{i\in J})$ into a lattice polytope (the existence is due to the rationality of b_i's), and D_J is the closed subvariety of $\mathbb{P}_{\Sigma/q}$ corresponding to the cone J in the quotient fan Σ/q. This can be further expressed as

$$|l \cdot E^{q,J}((b_i)_{i \in J}) \cap M| = \chi(\mathbb{P}_{\Sigma/(q \sqcup J)}, \mathcal{O}_{\mathbb{P}_{\Sigma/(q \sqcup J)}}(lD))$$

Step 2. Denote the canonical map from the smooth toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ to its coarse moduli space (which is a simplicial toric variety) $\mathbb{P}_{\Sigma}$ by π. We denote the line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$ defined by the same support function with D by $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}(D)$. Since in the definition of $\mathbb{P}_{\mathbf{\Sigma}}$ the additional data of a vector on each ray of the stacky fan is chosen to be the primitive generator, we know that the pushforward of $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}(D)$ is exactly $\mathcal{O}_{\mathbb{P}_{\Sigma}}(D)$. On the other hand, it is a well-known result (see e.g., [2, Definition 4.1, Example 8.1]) for a tame Deligne-Mumford stack $\mathcal{X}$ with coarse moduli space X, the canonical map $\pi : \mathcal{X} \to X$ is cohomologically affine. This implies that $H^i(\mathcal{X}, \mathcal{F})$ is equal to $H^i(X, \pi_* \mathcal{F})$ for all $i > 0$ and any coherent sheaf $\mathcal{F}$. Apply this fact to our situation, we get $\chi(\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}, \mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}}(lD)) = \chi(\mathbb{P}_{\Sigma/(q\sqcup J)}, \mathcal{O}_{\mathbb{P}_{\Sigma/(q\sqcup J)}}(lD))$. Thus, we have

$$\chi(\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}, \mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}}(lD)) = |l \cdot E^{q,J}((b_i)_{i\in J}) \cap M|$$

Step 3.

Then we apply Corollary 31, we obtain

$$\begin{aligned} |l \cdot E^{q,J}((b_i)_{i\in J}) \cap M| &= \sum_{\gamma\in \mathrm{Box}(\Sigma/(q\sqcup J))} \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_\gamma ch^c_\gamma(l\cdot \mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}}(lD)) \operatorname{Td}(\gamma) \\ &= \sum_{\gamma\in \mathrm{Box}(\Sigma/(q\sqcup J))} \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_\gamma e^{lD} \cdot \operatorname{Td}(\gamma) \end{aligned}$$

note that since $q \sqcup J$ is an interior cone (because it contains an interior cone $\sigma(c)$ as a subcone), the quotient fan $\Sigma/(q \sqcup J)$ is complete, hence $\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}$ is compact therefore K_0 and K_0^c (and the corresponding Chern characters) coincide.

The *affine volume*[4] $\operatorname{vol}_{\text{aff}} E^{q,J}((b_i)_{i\in J})$ is computed by

$$\begin{aligned} \operatorname{vol}_{\text{aff}} E^{q,J}((b_i)_{i\in J}) &= \lim_{l\to\infty} \frac{|l \cdot E^{q,J}((b_i)_{i\in J}) \cap M|}{l^{\dim E^{q,J}((b_i)_{i\in J})}} \\ &= \sum_{\gamma\in \mathrm{Box}(\Sigma/(q\sqcup J))} \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \lim_{l\to\infty} \int_\gamma \frac{e^{lD}}{l^{\mathrm{rk}N-1-|J|}} \cdot \operatorname{Td}(\gamma). \end{aligned}$$

In the last step, we used $\dim E^{q,J}((b_i)_{i\in J}) = \mathrm{rk}N - 1 - |J|$.

Now, we claim that the only nonzero term in this sum is the $\gamma = 0$ term. To see this, we expand the e^{lD} and the Todd class $\operatorname{Td}(\gamma)$ as sums of homogeneous components:

$$\int_\gamma \frac{e^{lD}}{l^{\mathrm{rk}N-1-|J|}} \cdot \operatorname{Td}(\gamma) = \int_\gamma \sum_{i,j=0}^{\infty} \frac{1}{l^{\mathrm{rk}N-1-|J|}} \frac{l^i D^i}{i!} \operatorname{Td}(\gamma)_j$$

[4]Note that the affine volume differs with the residual volume $\operatorname{vol} E^{q,J}((b_i)_{i\in J})$ by a factor of the index of b_i's, namely the index of the sublattice spanned by b_i's inside the standard lattice $\mathbb{Z}^d$, see step 4.

where $\mathrm{Td}(\gamma)_j$ denotes the degree j part of the Todd class of γ. By definition, the only nonzero contribution comes from terms with (i, j) such that

$$\deg(D^i) + \deg(\mathrm{Td}(\gamma)_j) = i + j$$

is exactly equal to $\dim(\gamma) = \mathrm{rk}N - 1 - |J| - |\sigma(\gamma)|$. On the other hand, if $i < \mathrm{rk}N - 1 - |J|$, the integral will be killed by taking limit $l \to \infty$ due to the factor of $\frac{1}{l^{\mathrm{rk}N-1-|J|}}$, thus $i \geq \mathrm{rk}N - 1 - |J|$. Combining these two observations, we can deduce that in order to have nonzero contribution, we must have

$$\mathrm{rk}N - 1 - |\sigma(\gamma)| - |J| = i + j \geq \mathrm{rk}N - 1 - |J| + j$$

which simplifies to $|\sigma(\gamma)| \leq -j$. This forces $j = 0$, $i = \mathrm{rk}N - 1 - |J|$ and $\sigma(\gamma) = \emptyset$, i.e, $\gamma = 0$. Hence the claim is proved. Therefore we have

$$\mathrm{vol}_{\mathrm{aff}}\, E^{q,J}((b_i)_{i\in J}) = \int_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}} \frac{D^{\mathrm{rk}N-1-|J|}}{(\mathrm{rk}N - 1 - |J|)!} = \int_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}} e^D$$

Step 4. The affine and residual volume of the polytope $E^{q,J}((b_i)_{i\in J})$ are related by the following equation:

$$\mathrm{vol}_{\mathrm{aff}}\, E^{q,J}((b_i)_{i\in J}) = (\text{index of } b_i's) \cdot \mathrm{vol}\, E^{q,J}((b_i)_{i\in J})$$

The index of b_i's is exactly equal to $|\,\mathrm{Box}(q \sqcup J)|$. Therefore, we have

$$\begin{aligned}
\mathrm{vol}\, E^{q,J}((b_i)_{i\in J}) &= \frac{1}{|\,\mathrm{Box}(q \sqcup J)|} \int_{\mathbb{P}_{\mathbf{\Sigma}/(q\sqcup J)}} e^D \\
&= \int_{\mathbb{P}_{\mathbf{\Sigma}}} e^D \cdot F_{q\sqcup J} \\
&= \frac{1}{|\,\mathrm{Box}(\sigma(\gamma))|} \int_{\mathbb{P}_{\mathbf{\Sigma}/\sigma(\gamma)}} e^D \cdot F_{q\sqcup J\setminus\sigma(\gamma)}
\end{aligned}$$

$$= \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_{\mathbb{P}_{\mathbf{\Sigma}/\sigma(\gamma)}} e^D \cdot \frac{D_{q \sqcup J}}{D_{\sigma(c)}} F_{I_c}$$

where the second equality follows from the fact that the ratio between the volume of a cone in the fan Σ and that of the corresponding quotient cone in the quotient fan $\Sigma/(q \sqcup J)$ is equal to $|\operatorname{Box}(q \sqcup J)|$. The third equality holds for similar reasons. The last equality is a consequence of the relations in the orbifold cohomology space. This concludes the proof.

4.4 Equality of A-brane and B-brane central charges

The goal of this section is to establish the equality between A-brane and B-brane central charges. This is accomplished by utilizing the hypergeometric duality [5] as a key ingredient. For readers' convenience, we briefly recall the statements here.

Definition 76. For any pair of solutions (Φ_c) and (Ψ_d) of the systems $\mathrm{bbGKZ}(C, 0)$ and $\mathrm{bbGKZ}(C^\circ, 0)$, we define a pairing

$$\langle -, - \rangle : \mathrm{Sol}(\mathrm{bbGKZ}(C, 0)) \times \mathrm{Sol}(\mathrm{bbGKZ}(C^\circ, 0)) \to \mathbb{C}$$

by the following formula

$$\langle \Phi, \Psi \rangle = \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I \left(\prod_{i \in I} x_i \right) \Phi_c \Psi_d$$

where the coefficient $\xi_{c,d,I}$ is defined as follows. Fix a choice of a generic vector $v \in C^\circ$. For a set I of size $\mathrm{rk} N$ we consider the cone $\sigma_I = \sum_{i \in I} \mathbb{R}_{\geq 0} v_i$. The coefficients $\xi_{c,d,I}$ for $c + d = v_I$ are defined as

$$\xi_{c,d,I} = \begin{cases} (-1)^{\deg(c)}, \text{ if } \dim \sigma_I = \mathrm{rk}\, N \text{ and both } c + \varepsilon v \text{ and } d - \varepsilon v \in \sigma_I^\circ \\ 0, \text{ otherwise.} \end{cases}$$

Here the condition needs to hold for all sufficiently small positive number $\varepsilon > 0$.

The main result in [5] states that this pairing is non-degenerate.

Theorem 77. *For any pair of solutions* (Φ_c) *and* (Ψ_d) *of the systems* $\mathrm{bbGKZ}(C,0)$ *and* $\mathrm{bbGKZ}(C^\circ,0)$*, the pairing* $\langle \Phi, \Psi\rangle$ *is a constant. Furthermore, the constant pairing* $\langle \Gamma, \Gamma^\circ\rangle$ *of the cohomology-valued Gamma series is equal to the inverse of the Euler characteristic pairing* $\chi : H^*_{\mathrm{orb}} \otimes H^*_{\mathrm{orb},c} \to \mathbb{C}$ *in the large radius limit. In particular,* $\langle -,-\rangle$ *is non-degenerate.*

We now combine the computation in Section 4.2 together with the hypergeometric duality to obtain the equality between A-brane and B-brane central charges. Specifically, we begin by proving the equality for the case of structure sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$ and its mirror cycle $\mathbb{R}^d_{\geq 0}$.

To start with, we recall the asymptotic behavior of the Gamma series Γ that was computed in [5].

Lemma 78. Let $t \to +\infty$, then for lattice point $c \in C$ and $\gamma \neq \gamma^\vee(c)$, the summand of $\Gamma_c(t^{-\psi(v_1)}x_1, \cdots, t^{-\psi(v_n)}x_n)$ is $o(t^{\psi(c)})$. For $\gamma = \gamma^\vee(c)$, we have

$$\Gamma_c(t^{-\psi(v_1)}x_1, \ldots) = t^{\psi(c)} \prod_{i=1}^n \mathrm{e}^{\frac{D_i}{2\pi \mathrm{i}}(\log x_i - \psi(v_i)\log t)} \prod_{i=1}^n \frac{x_i^{-c_i}}{\Gamma(1-c_i+\frac{D_i}{2\pi\mathrm{i}})}(1+o(1)).$$

Proof. See [5, Lemma 3.10]. □

Theorem 79. *The A-brane central charge associated to the positive real locus* $(\mathbb{R}_{\geq 0})^d$ *coincides with the B-brane central charge associated to the structure sheaf* $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$*.*

Proof. Throughout this proof, we will denote an interior lattice point by $d \in C^\circ$ and denote the rank of the lattice N by $\mathrm{rk}N$.

Consider the pairing

$$\langle \Gamma, Z^{A,\mathbb{R}^d_{>0}}\rangle = \sum_{c,d,I} \xi_{c,d,I}\mathrm{Vol}_I(\prod_{i\in I} x_i)\Gamma_c \cdot Z_d^{A,\mathbb{R}^d_{>0}} \in H^*_{\mathrm{orb}}(\mathbb{P}_{\mathbf{\Sigma}})$$

we look at the component corresponding to a fixed twisted sector γ. Combining Theorem 68, Lemma 78, by an argument similar to the proofs of [5, Proposition 3.12, 3.13], the asymptotic behavior of

$$\prod_{i=1}^{n}(t^{-\psi(v_i)})\Gamma_{c,\gamma}(t^{-\psi(v_i)})Z_d^{A,\mathbb{R}^d_{>0}}(t^{-\psi(v_i)})$$

is given by $o(1)$ unless $\gamma = \gamma^\vee(c) = \gamma(d)$ and both I_c and I_d are cones in Σ, in which case the asymptotic behavior is

$$o(1) + \frac{1}{(2\pi\mathrm{i})^{|I_c|}} \cdot \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \prod_{i=1}^{n} \mathrm{e}^{\frac{D_i}{2\pi\mathrm{i}}(-\psi(v_i)\log t)} \frac{(-1)^{\mathrm{rk}N-\deg d}}{(2\pi\mathrm{i})^{|\sigma(d)|}|\operatorname{Box}(\sigma(\gamma))|} \int_\gamma t^\omega \hat{\Gamma}_\gamma F_{I_d}.$$

Since $\langle \Gamma, \Psi \rangle$ is a constant, taking the constant term we get

$$\begin{aligned}
\langle \Gamma, \Psi \rangle_\gamma &= \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \frac{(-1)^{\mathrm{rk}N-\deg d}}{(2\pi\mathrm{i})^{|I_c|+|\sigma(d)|}|\operatorname{Box}(\sigma(\gamma))|} \int_\gamma \hat{\Gamma}_\gamma F_{I_d} \\
&= \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \frac{(-1)^{\mathrm{rk}N-\deg d}}{(2\pi\mathrm{i})^{\mathrm{rk}N}|\operatorname{Box}(\sigma(\gamma))|} \int_{\gamma^\vee} (-1)^{\dim\gamma^\vee - |I_d|} (\hat{\Gamma}_\gamma)^* F_{I_d} \\
&= \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \frac{(-1)^{\mathrm{rk}N-\deg d}}{(2\pi\mathrm{i})^{\mathrm{rk}N}|\operatorname{Box}(\sigma(\gamma))|} \\
&\qquad \cdot \int_{\gamma^\vee} (2\pi\mathrm{i})^{|\sigma(\gamma)|} (-1)^{\deg\gamma^\vee + \dim\gamma^\vee - |I_d|} \frac{F_{I_d}}{\hat{\Gamma}_{\gamma^\vee}} \mathrm{Td}(\gamma^\vee) \\
&= \frac{1}{(2\pi\mathrm{i})^{\mathrm{rk}N}} \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I (2\pi\mathrm{i})^{|\sigma(\gamma)|} \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \frac{1}{|\operatorname{Box}(\sigma(\gamma))|} \int_{\gamma^\vee} \frac{F_{I_d}}{\hat{\Gamma}_{\gamma^\vee}} \mathrm{Td}(\gamma^\vee) \\
&= \frac{1}{(2\pi\mathrm{i})^{\mathrm{rk}N}} \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I (2\pi\mathrm{i})^{|\sigma(\gamma)|} \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \cdot (1, \frac{F_{I_d}}{\hat{\Gamma}_{\gamma^\vee}})_{\mathrm{orb},\gamma}
\end{aligned}$$

Here we used $\deg\gamma^\vee = |\sigma(\gamma)| + |I_d| - \deg d = |\sigma(d)| - \deg d$, therefore

$$\begin{aligned}
(-1)^{\mathrm{rk}N - \deg d + \deg\gamma^\vee + \dim\gamma^\vee - |I_d|} &= (-1)^{\mathrm{rk}N + |\sigma(d)| + \mathrm{rk}N - |\sigma(\gamma)| - |I_d|} \\
&= (-1)^{\mathrm{rk}N + |\sigma(d)| + \mathrm{rk}N - |\sigma(d)|} = 1
\end{aligned}$$

By [5, Theorem 4.2] the following class in $H^*_{\mathrm{orb}} \otimes H^*_{\mathrm{orb},c}$

$$\frac{1}{(2\pi \mathrm{i})^{\mathrm{rk}N}} \bigoplus_{\gamma} \sum_{c,d,I} \xi_{c,d,I} \mathrm{Vol}_I (2\pi \mathrm{i})^{|\sigma(\gamma)|} \frac{D_{I_c}}{\hat{\Gamma}_\gamma} \otimes \frac{F_{I_d}}{\hat{\Gamma}_{\gamma^\vee}}$$

is inverse to the Euler characteristic pairing, therefore for any γ we have

$$\langle \Gamma, Z^{A,\mathbb{R}^d_{>0}} \rangle_\gamma = 1_\gamma$$

so $\langle \Gamma, Z^{A,\mathbb{R}^d_{>0}} \rangle = \bigoplus_\gamma 1_\gamma = ch(\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}})$, i.e., $Z^{A,\mathbb{R}^d_{>0}}$ corresponds to the structure sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$. □

We have completed the proof for the case of structure sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}$. Next, we consider an arbitrary line bundle $\mathcal{L} = \mathcal{O}(\sum_{i=1}^n a_i D_i)$ corresponding to a torus-invariant divisor $\sum_{i=1}^n a_i D_i$. The mirror cycle $\mathrm{mir}(\mathcal{L})$ of $\mathcal{L}$ is constructed from $\mathbb{R}^d_{>0}$ by monodromy. More precisely, the divisor $\sum_{i=1}^n a_i D_i$ defines a loop in the stringy Kähler moduli space of $\mathbb{P}_{\mathbf{\Sigma}}$ by

$$\phi : [0,1] \to \mathbb{C}^n, \quad \theta \mapsto (e^{-2\pi \mathrm{i} a_1 \theta}, \cdots, e^{-2\pi \mathrm{i} a_n \theta}) \tag{4.4.1}$$

We denote the Laurent polynomial corresponding to $\phi(\theta)$ by $f^{(\theta)}$, then we have a family of hypersurfaces $Z_{f^{(\theta)}}$ in $(\mathbb{C}^*)^d$, where $Z_{f^{(1)}} = Z_{f^{(0)}} = Z_f$. We then define the mirror cycle of $\mathcal{L}$ to be the parallel transport of $\mathbb{R}^d_{>0}$ along this loop.

Corollary 80. For any $\mathcal{L} = \mathcal{O}(\sum_{i=1}^n a_i D_i) \in K_0(\mathbb{P}_{\mathbf{\Sigma}})$, the A- and B-brane central charges coincide:

$$Z^{A,\mathrm{mir}(\mathcal{L})} = Z^{B,\mathcal{L}}.$$

Proof. It suffices to compare the monodromy along the loop (4.4.1) on both sides.

Recall that the Gamma series is given by

$$\Gamma_c^\circ(x_1,\ldots,x_n) = \bigoplus_\gamma \sum_{l\in L_{c,\gamma}} \prod_{i=1}^n \frac{x_i^{l_i+\frac{D_i}{2\pi\mathrm{i}}}}{\Gamma(1+l_i+\frac{D_i}{2\pi\mathrm{i}})} \left(\prod_{i\in\sigma} D_i^{-1}\right) F_\sigma$$

and the monodromy comes from the term $\prod_{i=1}^n x_i^{l_i+\frac{D_i}{2\pi\mathrm{i}}} = \prod_{i=1}^n e^{(l_i+\frac{D_i}{2\pi\mathrm{i}})\log x_i}$.

Fix c and γ, when θ goes from 0 to 1, the x_i goes around the origin a_i times clockwisely and hence the original $\log x_i$ now becomes $\log x_i - 2\pi\mathrm{i}a_i$, therefore contributes an extra factor $e^{-a_i(2\pi\mathrm{i}l_i+D_i)}$. Take product over all $i=1,\cdots,n$, this is $e^{-\sum_i a_i(2\pi\mathrm{i}l_i+D_i)}$. By definition of $l\in L_{c,\gamma}$, we have $l_i \equiv \gamma_i \mod \mathbb{Z}$, therefore the factor is equal to $e^{-\sum_i a_i(2\pi\mathrm{i}\gamma_i+D_i)}$, which is exactly the Chern character $ch_\gamma(\mathcal{O}(-\sum_{i=1}^n a_iD_i))$.

Consequently, the effect of the monodromy on the Gamma series is to multiply it by $ch(\mathcal{O}(-\sum_{i=1}^n a_iD_i))$. Then the B-brane central charge is obtained by composing with $\chi(ch(\mathcal{O}_{\mathbb{P}_{\mathbf{\Sigma}}}),-)$, which is

$$\chi\left(1, ch(\mathcal{O}(-\sum_{i=1}^n a_iD_i))\cdot\Gamma_c^\circ\right) = \chi\left(ch(\mathcal{O}(\sum_{i=1}^n a_iD_i)),\Gamma_c^\circ\right)$$

by Proposition 30. This is exactly the central charge $Z^{B,\mathcal{O}(\sum_{i=1}^n a_iD_i)}$. It then follows directly from the construction of the mirror cycle of $\mathcal{O}(\sum_{i=1}^n a_iD_i)$ that the monodromy on the A-side matches with the monodromy on the B-side. □

The proof of the second main result (Theorem 62) of this dissertation is now completed.

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