GKZ Hypergeometric Systems and Their Applications to Mirror Symmetry

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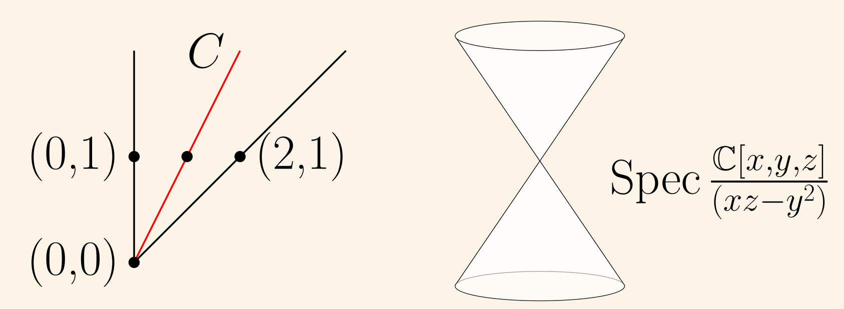
Motivation

Consider a $(d+1)$-dimensional affine toric variety $\mathrm{Spec}(\mathbb{C}[C \cap \mathbb{Z}^{d+1}])$ with Gorenstein singularities. A triangulations Σ of the cone C gives a (crepant) resolution $\mathbb{P}_\Sigma$ of the singularities.

Example: A_1-singularity



Any two crepant resolutions $\mathbb{P}_{\Sigma_1}$ and $\mathbb{P}_{\Sigma_2}$ are derived equivalent:

$$D^b(\mathbb{P}_{\Sigma_1}) \xrightarrow{\sim} D^b(\mathbb{P}_{\Sigma_2})$$

According to homological mirror symmetry, there should exist an **isotrivial family of triangulated categories** over the stringy Kähler moduli space.

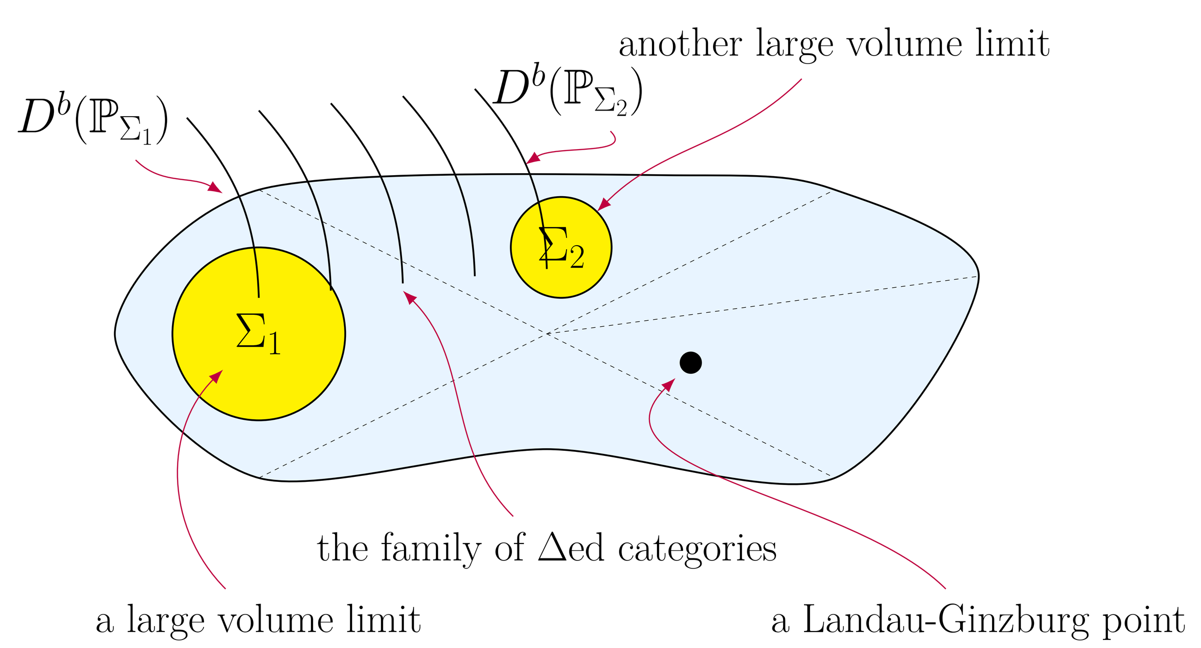


Figure 1: stringy Kähler moduli space

It's a very difficult problem to construct such a family of triangulated categories, so we look at their Grothendieck groups (K-theories).

isotrivial family of triangulated categories (very difficult) → de-categorification → GKZ systems (well-understood)

GKZ systems

For each lattice point c in the cone C we attach a holomorphic function $\Phi_c(x_1, \dots, x_n)$ defined on the stringy Kähler moduli space, and consider a linear system of PDEs:

$$\mathrm{bbGKZ}(C,0): \begin{cases} \partial_i \Phi_c = \Phi_{c+v_i}, & \forall c \in C, i = 1, \cdots, n \\ \sum_{i=1}^n \langle \mu, v_i \rangle x_i \partial_i \Phi_c + \langle \mu, c \rangle \Phi_c = 0, & \forall c \in C, \mu \in N^\vee \end{cases}$$

if we only use lattice points in the interior C°, we get a compactly-supported version $\mathrm{bbGKZ}(C^\circ, 0)$.

Their **solution spaces** are naturally identified with the **Grothendieck groups** of the usual derived category $D^b(\mathbb{P}_\Sigma)$ and the compactly-supported $D^b_c(\mathbb{P}_\Sigma)$ via some **hypergeometric series**.

Example

In the case of A_1-singularity, the system $\mathrm{bbGKZ}(C,0)$ reduces to two equations on a single function $\Phi = \Phi_{(0,0)}(x_1, x_2, x_3)$:

$$(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\Phi = 0, \quad (x_2\partial_2 + 2x_3\partial_3)\Phi = 0,$$

The solution space to this system is isomorphic to the K_0 of the blowup of $\mathrm{Spec}\,\frac{\mathbb{C}[x,y,z]}{(xz-y^2)}$ at the origin, which is 2-dimensional.

Duality

There is an Euler pairing between the derived categories $D^b(\mathbb{P}_\Sigma)$ and $D^b_c(\mathbb{P}_\Sigma)$

$$\chi : D^b(\mathbb{P}_\Sigma) \times D^b_c(\mathbb{P}_\Sigma) \longrightarrow \mathbb{Z}, \quad (\mathcal{F}^\bullet, \mathcal{G}^\bullet) \mapsto \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(\mathcal{G}^\bullet, \mathcal{F}^\bullet)$$

which descends to the level of K-theories. In [1] we find a formula for this pairing in terms of solutions to GKZ systems.

Theorem (Borisov-H.[1])

Let Φ and Ψ be solutions to $\mathrm{bbGKZ}(C,0)$ and $\mathrm{bbGKZ}(C^\circ,0)$ respectively. We define the **GKZ pairing**

$$\langle \Phi, \Psi \rangle_{\mathrm{GKZ}} = \sum_{\substack{c \in C, d \in C^\circ \\ I \subseteq \{1,\cdots,n\}, |I| = \mathrm{rk} N}} \xi_{c,d,I} \,\mathrm{Vol}_I \left(\prod_{i \in I} x_i \right) \Phi_c \Psi_d$$

here $\xi_{c,d,I} = 0, \pm 1$ is determined by the combinatorics of C, and Vol_I denotes the volume of the cone generated by I. Then $\langle -, - \rangle_{\mathrm{GKZ}}$ agrees with the Euler pairing $\chi(-,-)$, in the neighborhood of any large volume limit Σ.

This formula is inspired by the (cohomological) formula of resolution of diagonal of toric varieties of Fulton-Sturmfels and the Gamma integral structure of Iritani.

Analytic continuation = Fourier-Mukai

We consider different triangulations Σ_1 and Σ_2. We can analytically continue the solutions defined on a neighborhood of Σ_1 to a neighborhood of Σ_2:

$$\{\text{sol. near } \Sigma_1\} \xrightarrow{a.c.} \{\text{sol. near } \Sigma_2\}$$

In [2] we proved that the operation of analytic continuation is realized by a natural Fourier-Mukai transform $K_0(\mathbb{P}_{\Sigma_1}) \to K_0(\mathbb{P}_{\Sigma_2})$.

References

[1] Lev Borisov and Zengrui Han. On hypergeometric duality conjecture. *Advances in Mathematics*, 442:109582, 2024.

[2] Zengrui Han. Analytic continuation of better-behaved GKZ systems and Fourier-Mukai transforms. *arXiv:2305.12241*, 2023.

[3] Zengrui Han. Central charges in local mirror symmetry via hypergeometric duality. *arXiv:2404.16258*, 2024.

Applications to mirror symmetry: equality of A- and B- central charges

Mirror symmetry for toric Calabi-Yau:

	A-side	B-side
spaces	Laurent polynomial $f : (\mathbb{C}^*)^d \to \mathbb{C}$	toric CY orbifolds $\mathbb{P}_\Sigma$
(A- or B-)branes	certain Lagrangian submanifolds L of $(\mathbb{C}^*)^d$	coherent sheaves on $\mathbb{P}_\Sigma$
categories of branes	Fukaya-Seidel category $FS((\mathbb{C}^*)^d, f)$	derived category $D^b(\mathbb{P}_\Sigma)$
central charges	period integrals	hypergeometric series

Example

In the case of A_1-singularity, the A-brane central charge looks like

$$\int_0^\infty \frac{dz}{x_1 + x_2 z + x_3 z^2}$$

and the B-brane central charge looks like

$$\sum_{i=0}^\infty \frac{(2i)!}{(i!)^2} (x_1 x_2^{-2} x_3)^i \cdot \left(\log(x_1 x_2^{-2} x_3) + \text{ higher order terms. } \right)$$

The GKZ systems provide tools to connect central charges on the two sides.

A-side (LG model) ⟷ GKZ systems ⟷ B-side (toric CY)

Theorem (H.[3])

The A-brane and B-brane central charges are identified under the (conjectural) homological mirror symmetry equivalence $FS((\mathbb{C}^*)^d, f) \xrightarrow{\sim} D^b(\mathbb{P}_\Sigma)$.

1. The leading term of the period integral is controlled by the tropical geometry of the Laurent polynomial f and is computable. Hence we can identify the leading terms of the A- and B-brane central charges.

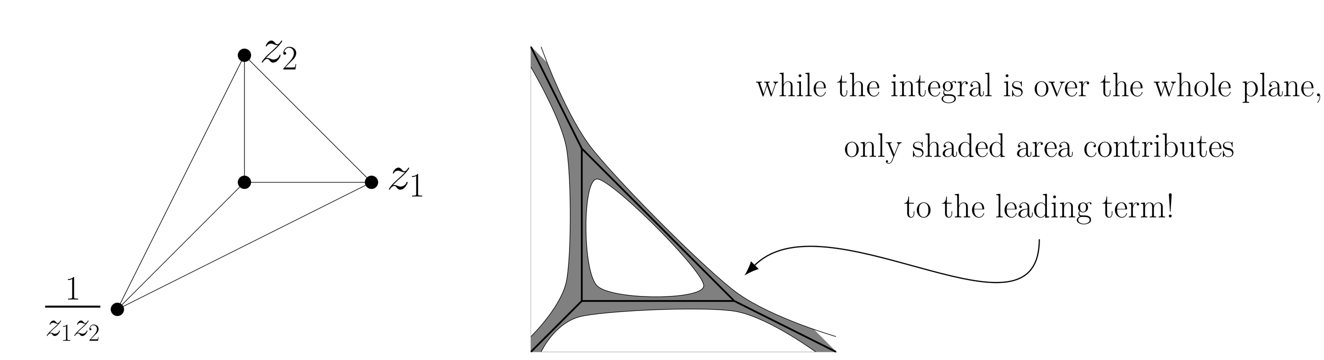


Figure 2: Newton polytope and tropical amoeba of $f = z_1 + z_2 + \frac{1}{z_1 z_2}$

2. Apply the hypergeometric duality to lift the equality of leading terms to the equality of the whole functions.