Stringy Hodge numbers of Pfaffian double mirrors and HPD

Zengrui Han (Rutgers University)

Motivation

The first example of derived equivalent but non-birationally equivalent Calabi-Yau 3-folds is the so-called **Pfaffian-Grassmannian correspondence**, studied by Borisov-Căldăraru [1] and Kuznetsov [2].

The Pfaffian double mirror construction is a higher-dimensional generaliztion. In this project we aim to study the relationship between two aspects of this construction: the **classical aspect** (Hodge numbers) and the **homological aspect** (derived categories).

Pfaffian double mirrors

Let V be a complex vector space of dimension $n \geq 4$. For $k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor - 1$, we define the **Pfaffian variety** $\mathrm{Pf}(2k, V)$ as

$$\mathrm{Pf}(2k, V) = \{\text{skew forms } \omega \text{ on } V \text{ with } \mathrm{rk}(\omega) \leq 2k\} \subseteq \mathbb{P}(\wedge^2 V^\vee)$$

Definition: Pfaffian double mirrors

Fix a generic subspace W of $\wedge^2 V^\vee$ of dimension l. We define

- X_W to be the complete intersection of $\mathbb{P}W^\perp$ and the Pfaffian variety $\mathrm{Pf}(2k, V^\vee)$ in $\mathbb{P}(\wedge^2 V)$.
- Y_W to be the complete intersection of $\mathbb{P}W$ and the Pfaffian variety $\mathrm{Pf}(2\lfloor \frac{n}{2} \rfloor - 2k, V)$ in $\mathbb{P}(\wedge^2 V^\vee)$.

When n is odd and $l = nk$, both X_W and Y_W are Calabi-Yau and they share the same mirror family, hence the name "double mirror".

Generally speaking the Pfaffian varieties and the linear sections X_W and Y_W are highly singular, therefore we need to replace them by certain categorical crepant resolutions.

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Stringy Hodge numbers

Batyrev introduced the notion of stringy E-function to define Hodge numbers for singular varieties.

Definition: Stringy E-function

Let X be a singular variety with at worst log-terminal singularities. Let $\pi : \widehat{X} \to X$ be a log resolution and $\{D_i\}_{i \in I}$ be the set of exceptional divisors. Denote $D_J^\circ = (\bigcap_{j \in J} D_j) \backslash (\bigcup_{j \notin J} D_j)$. The stringy E-function of X is defined as

$$E_{\mathrm{st}}(X; u, v) := \sum_{J \subseteq I} E(D_J^\circ) \prod_{j \in J} \frac{uv - 1}{(uv)^{\alpha_j + 1} - 1}$$

where α_j denotes the discrepancy of the exceptional divisor D_j, defined by the equation $K_{\widehat{X}} = \pi^* K_X + \sum_j \alpha_j D_j$. The definition is independent of the choice of π.

If $E_{\mathrm{st}}(X; u, v)$ is a polynomial, then we can use its coefficients to define Hodge numbers of X. However this is not true for Pfaffian varieties when n is even.

To address this issue, we introduce a modified version of stringy E-function, obtained by modifying the discrepancies of the log resolution we used for computation.

Theorem (H.[3])

Let X_W and Y_W be the Pfaffian double mirror pair. We have the following relation between their E-functions:

$$q^{2k\lfloor \frac{n}{2} \rfloor} E_{\mathrm{st}}(Y_W) - q^l E_{\mathrm{st}}(X_W) = \frac{q^l - q^{2k\lfloor \frac{n}{2} \rfloor}}{q - 1} \binom{\lfloor \frac{n}{2} \rfloor}{k}_{q^2}$$

where $q = uv$, and E_{st} is the stringy E-function of Batyrev when n is odd, and the modified E-function when n is even.

References

[1] Lev Borisov and Andrei Căldăraru. The Pfaffian-Grassmannian derived equivalence. *J. Algebraic Geom.*, 18(2):201–222, 2009.

[2] Alexander Kuznetsov. Homological projective duality. *Publ. Math. Inst. Hautes Études Sci.*, (105):157–220, 2007.

[3] Zengrui Han. Stringy Hodge numbers of Pfaffian double mirrors and Homological Projective Duality. *arXiv:2409.17449*, 2024.

[4] Jørgen Vold Rennemo and Ed Segal. Hori-mological projective duality. *Duke Math. J.*, 168(11):2127–2205, 2019.

Homological Projective Duality (HPD)

- We want to decompose the categorical resolution into smaller "blocks":

$$\widetilde{D}^b(\mathrm{Pf}(2k, V^\vee)) = \left\langle \mathcal{A}_0, \mathcal{A}_1(1), ..., \mathcal{A}_{i-1}(i-1) \right\rangle$$

where $\mathcal{A}_0 \supseteq \cdots \supseteq \mathcal{A}_{i-1}$ are admissible subcategories.
Similar to the classical Lefschetz hyperplane theorem, the categorical resolution of the linear sections consists of a **primitive part** $\mathcal{C}_W$ and **ambient part** comes from the Pfaffian:

$$\widetilde{D}^b(X_W) = \left\langle \mathcal{C}_W, \mathcal{A}_l(1), ..., \mathcal{A}_{i-1}(i-l) \right\rangle$$

- And there should exist a dual picture for the dual Pfaffian $\mathrm{Pf}(2\lfloor n/2 \rfloor - 2k, V)$ and its linear sections Y_W:

$$\widetilde{D}^b(\mathrm{Pf}(2\lfloor n/2 \rfloor - 2k, V)) = \left\langle \mathcal{B}_{j-1}(-j+1), ..., \mathcal{B}_1(-1), \mathcal{B}_0 \right\rangle$$

and

$$\widetilde{D}^b(Y_W) = \left\langle \mathcal{B}_{j-1}(N-l-j), ..., \mathcal{B}_{N-l}(-1), \mathcal{C}_W \right\rangle$$

- The main point of HPD is that the primitive part $\mathcal{C}_W$ in the two decompositions are the **same**.

This picture has been established by Rennemo and Segal [4] rigorously for odd-dimensional Pfaffians, while it is still open for even dimensional cases. The key question is the following.

Question

What do the blocks $\mathcal{A}$ and $\mathcal{B}$ look like when n is even?

We provide a partial answer on the level of Grothendieck groups.

Conjecture (H.[3])

For $\mathrm{Pf}(2k, V^\vee)$:

- $\mathcal{A}_0 = \cdots = \mathcal{A}_{nk - \frac{n}{2} - 1}$ are $nk - \frac{n}{2}$ blocks of size $\binom{n/2}{k}$.
- $\mathcal{A}_{nk - \frac{n}{2}} = \cdots = \mathcal{A}_{nk-1}$ are $n/2$ blocks of size $\binom{n/2-1}{k}$.

Similarly, for the dual $\mathrm{Pf}(2\lfloor n/2 \rfloor - 2k, V)$:

- $\mathcal{B}_0 = \cdots = \mathcal{B}_{\frac{n(n-1)}{2} - nk - 1}$ are $\frac{n(n-1)}{2} - nk$ blocks of size $\binom{n/2}{n/2-k}$.
- $\mathcal{B}_{\frac{n(n-1)}{2} - nk} = \cdots = \mathcal{B}_{\frac{n^2}{2} - nk - 1}$ are $n/2$ blocks of size $\binom{n/2-1}{n/2-k}$.

It is an ongoing project jointly with Lev Borisov and Kimoi Kemboi (Princeton) to prove the even-dimensional Pfaffian HPD.